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Part II. Partial Differential Equations of the Second Order.

1. The Cauchy-Kowalewski Theorem. As in Part I, our work depends directly on the Cauchy-Kowalewski theorem for equations of the second order.* Let us consider a partial differential equation of the second order in one dependent and two independent variables,

$$F[x, y, z, p, q, r, s, t] = 0 \tag{1}$$

where $p = \partial z/\partial x$, $q = \partial z/\partial y$, $r = \partial^2 z/\partial x^2$, $s = \partial^2 z/\partial x \partial y$, $t = \partial^2 z/\partial y^2$; and let this equation be solved for r,

$$r = f[x, y, z, p, q, s, t]. \tag{2}$$

THEOREM: *Then there exists one and only one solution*

$$z = v(x, y)$$

which satisfies the boundary condition,

$$z\Big]_{x=x_0} = \phi(y), \qquad p\Big]_{x=x_0} = \psi(y) \tag{3}$$

where $\phi(y)$ and $\psi(y)$ are each preassigned analytic functions of y near $y = y_0$; provided that $f[x, y, z, p, q, s, t]$ is a single valued analytic function of x, y, z, p, q, s, t near the point $x = x_0$, $y = y_0$, $z = z_0 = \phi(y_0)$, $p = p_0 = \psi(y_0)$, $q = q_0 = \phi'(y_0)$, $s = s_0 = \psi'(y_0)$, $t = t_0 = \phi''(y_0)$, and $v(x, y)$ is analytic near (x_0, y_0); where $\phi'(y) = d\phi(y)/dy$, $\phi''(y) = d^2\phi(y)/dy^2$, etc. The equation (2) is said to be in *normal form*, if this condition is satisfied. The equation (1) can be put in the form (2) where f (x, y, z, p, q, s, t) is analytic at the point specified, provided $F(x, y, z, p, q, r, s, t)$ is analytic, and $\partial F/\partial r \neq 0$, at the point $(x_0, y_0, z_0, p_0, q_0, r_0, s_0, t_0)$, where $F[x_0, y_0, z_0, p_0, q_0, r_0, s_0, t_0] = 0$.

Geometrically the theorem states that one and only one analytic integral surface of (1) (corresponding to each single valued solution of the form (2)) exists, upon which lies a given *strip* (curve with appended tangent planes) of which the curve lies in a plane parallel to the plane of YOZ, provided only that F is analytic along the given strip in the manner specified above.

Following Part I, we will then try to find those *strips* in space for which

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* See footnotes, p. 121.

the equation (1) cannot be put into normal form, and for which, for this reason, the Cauchy-Kowalewski theorem may fail.

2. Characteristic Strips on a given Integral Surface. Let there be given an integral surface

$$z = \nu(x, y) \tag{4}$$

where $\nu(x, y)$ is a known analytic function of x and y. We will first determine those strips which lie on this surface, and for which the Cauchy-Kowalewski theorem may fail. Let $y = \lambda(x)$ be the equation of a curve on the surface (4) where $\lambda(x)$ is an analytic function of x. The strip determined by this curve, and the tangent planes to the surface (4) at each point of the curve is then given by the equations

$$(5)\quad \begin{cases} y = \lambda(x) \\ z = \nu(x, y) \\ p = \dfrac{\partial \nu(x, y)}{\partial x} \\ q = \dfrac{\partial \nu(x, y)}{\partial y} \end{cases} \quad \text{or} \quad \begin{cases} y = \lambda(x) \\ z = \nu(x, \lambda(x)) & = \mu(x) \\ p = \dfrac{\partial \nu(x, y)}{\partial x}\Big]_{y=\lambda(x)} & = \phi_1(x) \\ q = \dfrac{\partial \nu(x, y)}{\partial y}\Big]_{y=\lambda(x)} & = \phi_2(x). \end{cases}$$

Let us now transform the equation (1) by the transformation

$$(6)\quad \begin{cases} x_1 = y - \lambda(x) \\ y_1 = x \end{cases} \quad \text{or} \quad \begin{cases} x = y_1 \\ y = x_1 + \lambda(y_1) \end{cases}$$

by means of which p and q are transformed as follows

$$(7)\quad \begin{cases} p_1 = q \\ q_1 = q\lambda'(y_1) + p \end{cases} \quad \text{or} \quad \begin{cases} p = p_1\lambda(x) + q_1 \\ q = p_1 \end{cases}$$

where $\lambda'(x) = d\lambda(x)/dx$; and the strip (5) goes into the strip

$$(8)\quad \begin{cases} x_1 = 0 & p_1 = \phi_2(y_1) \\ z = \mu(y_1) & q_1 = \phi_2(y_1)\lambda'(y_1) + \phi_1(y_1) = \mu'(y_1) \end{cases}$$

which is of the form (3) to which the Cauchy-Kowalewski theorem applies,

where $\mu(y_1)$ and $\phi_2(y_1)$ are any analytic functions of y_1, and $\mu'(y_1) = d\mu(y_1)/dy_1$. The second derivatives r_1, s_1, t_1 are given by the equations:

$$(9)\quad \begin{cases} r_1 = \dfrac{\partial^2 z}{\partial x^2} = t \\ s_1 = \dfrac{\partial^2 z}{\partial x_1\, \partial y_1} = s + t\lambda'(y_1) \\ t_1 = \dfrac{\partial^2 z}{\partial y_1^2} = r + 2s\lambda' + t\lambda'^2 + q\lambda'' \end{cases} \quad \text{or} \quad \begin{cases} r = r_1\lambda'^2 - 2s_1\lambda' + t_1 - p_1\lambda'' \\ s = s_1 - r_1\lambda' \\ t = r_1 \end{cases}$$

where $\lambda' = d\lambda(x)/dx = d\lambda(y_1)/dy_1$, etc.: and hence the transformed differential equation is

$$(10)\quad F[y_1; x_1 + \lambda(y_1); z; q_1 + p_1\lambda; p_1; r_1\lambda'^2 - 2s_1\lambda' + t_1 - p_1\lambda''; s_1 - r_1\lambda'; r_1] = 0.$$

Hence if F is analytic and $\partial F/\partial r_1 \neq 0$ near the point $x_1 = 0$, $y_1 = y_1'$, $z = \mu(y_1)$, $p_1 = \phi_2(y_1)$, $q_1 = \mu'(y_1)$, $r_1 = \chi(y_1)$, $s_1 = \phi_2'(y_1)$, $t_1 = \mu''(y_1)$, where $\chi(y_1)$ is such that (10) is satisfied, then the strip (5), by the theorem, determines uniquely an analytic integral surface. Or, as in §2, Part I, we may transform back into the old variables and substitute $v(x, y)$ throughout for z, and then $\lambda(x)$ for y, which will be equivalent to the above substitution since (4) is a solution of (1), and since (7) and (9) give the same values for p, q, r, s, t as are obtained from $p = v_x[x, \lambda(x)]$, $q = v_y[x, \lambda(x)]$, $r = v_{xx}[x, \lambda(x)]$, etc.

But if

$$(11)\quad \frac{\partial F}{\partial r_1} \equiv \frac{\partial F}{\partial r}\left[\frac{d\lambda(x)}{dx}\right]^2 - \frac{\partial F}{\partial s}\,\frac{d\lambda(x)}{dx} + \frac{\partial F}{\partial t} = 0$$

when $v(x, y)$ is substituted for z, and then $\lambda(x)$ is substituted for y, the Cauchy-Kowalewski theorem may fail for the strip (5). If then $v(x, y)$ is substituted throughout for z in (11), this equation becomes an ordinary differential equation of the first order and second degree for y as a function of x,

$$(11a)\quad R(x, y)\left(\frac{dy}{dx}\right)^2 - S(x, y)\,\frac{dy}{dx} + T(x, y) = 0,$$

where R, S, T are the functions of x and y obtained from $\partial F/\partial r$, $\partial F/\partial s$, $\partial F/\partial t$ respectively, when $v(x, y)$ is substituted for z. Any solution of this

equation $y = \lambda(x)$ defines a strip (5) for which the Cauchy-Kowalewski theorem may fail. Any curve

$$(12) \qquad \begin{cases} y = \lambda(x) \\ z = \nu(x, y) \end{cases}$$

which lies on an integral surface (4) will be called a *characteristic curve*, if $y = \lambda(x)$ is a solution of $(11a)$.* The corresponding strip(5) will be called a *characteristic strip*, where the tangent plane attached to the curve at each point is the tangent plane of the surface (4) at that point.

It is clear that there will be two one parameter families of characteristic curves, in general, on every integral surface, of which two curves and but two will pass through any given point of the surface. These two one parameter families correspond to the two solutions of $(11a)$, as an algebraic equation for dy/dx. The two families will be both *real*, both *imaginary*, or *coincident*, according as $S^2 - 4RT > 0, < 0, = 0$, on the surface (4). The equation (1) is called a *hyperbolic*, *elliptic*, or *parabolic* equation on the surface (4), or in any region upon it, in these three cases respectively.

Any other curve of the type (12) on the surface (4) together with its strip determined by (5) evidently determines the integral surface (4) uniquely, provided F is analytic along the strip.

We have tacitly assumed that $R \not\equiv 0$ identically in x and y on the surface (4). If $R = 0$, we can evidently do the same work by interchanging x and y throughout unless T also vanishes identically on the surface (4), or by replacing x and y by $x - y$ and $x + y$ respectively, unless $S \equiv 0$ on the surface(4). If $R \equiv 0$, $T \equiv 0$, and $S \equiv 0$ (so that $RT - S^2 \equiv 0$) on the surface (4), we will call (4) a *singular integral surface*,† and we will exclude such a surface explicitly in the statement of all our theorems, for it is evident that any curve on such a surface would satisfy the equation $(11a)$, and that for it the Cauchy-Kowalewski theorem might fail.

It is evident that the same equation $(11a)$ after dy/dx is replaced by $[dx/dy]^{-1}$ determines whether a curve of the form

$$(13) \qquad \begin{cases} x = x_0 \\ z = \nu(x, y) \end{cases} \quad \text{or} \quad \begin{cases} x = x_0 \\ z = \nu(x_0, y) = \mu(y) \end{cases}$$

* Equation $(11a)$ will be recognized as the principal equation of the characteristics, as ordinarily derived. See Sommerfeld, *Ency. d. Math. Wiss.*, II, A, 7, p. 598; von Weber, *l. c.*, p. 366; Goursat, *l. c.*, *2 ordre*, I, p. 172.

† Compare p. 127, Art. 2; and see Darboux, *l. c.*

together with its corresponding strip, satisfies the requirements of the Cauchy-Kowalewski theorem, if F is analytic along the strip. For the requirement is precisely that $R \neq 0$ along the strip, and the equation (11a) reduces to this form for a curve of the above type.

3. The Characteristic Strips in Space. If the equation of the integral surface (4) is unknown, we shall have to determine $\mu(x)$, $\phi_1(x)$, $\phi_2(x)$ as well as $\lambda(x)$, in order to determine the characteristic strip (5). The values of y, z, p, q, r, s, t, as functions of x which satisfy (11) will then be given by $y=\lambda(x)$, $z=\mu(x)$, $p=\phi_1(x)$, $q=\phi_2(x)$, $r=\phi_3(x)$, $s=\phi_4(x)$, $t=\phi_5(x)$, say, for which the following several relations (13) must hold. First,

$$F[x, \lambda(x), \mu(x), \phi_1(x), \phi_2(x), \phi_3(x), \phi_4(x), \phi_5(x)] = 0 \tag{13a}$$

since the quantities substituted in (11) are to satisfy (1). Moreover since $v(x,y)$ is a solution of (1) we have

$$F[x\,;\ y\,;\ v(x, y)\,;\ v_x\,;\ v_y\,;\ v_{xx}\,;\ v_{xy}\,;\ v_{yy}] = 0 \tag{13b}$$

identically in x and y. Hence the total derivatives of F with respect to x and y formed from this equation vanish identically in x and y. If we introduce the notation

$$\begin{cases} X = \dfrac{\partial F}{\partial x} + p\dfrac{\partial F}{\partial z} + r\dfrac{\partial F}{\partial p} + s\dfrac{\partial F}{\partial q} \\ Y = \dfrac{\partial F}{\partial y} + q\dfrac{\partial F}{\partial z} + s\dfrac{\partial F}{\partial p} + t\dfrac{\partial F}{\partial q} \end{cases} \tag{13c}$$

we may write these relations as follows:

$$\begin{cases} \dfrac{dF}{dx} = X + \alpha R + \beta S + \gamma T = 0 \\ \dfrac{dF}{dy} = Y + \beta R + \gamma S + \delta T = 0 \end{cases} \tag{13d}$$

where $\alpha = \partial^3 z/\partial x^3$, $\beta = \partial^3 z/\partial x^2 \partial y$, $\gamma = \partial^3 z/\partial x \partial y^2$, $\delta = \partial^3 z/\partial y^3$. We have also the identities

$$\begin{cases} z' = p + qy' & r' = \alpha + \beta y' \\ p' = r + sy' & s' = \beta + \gamma y' \\ q' = s + ty' & t' = \gamma + \delta y' \end{cases} \tag{13e}$$

where $y' = dy/dx$, $z' = dz/dx$, etc., which are identically satisfied in x, when $v(x, y)$ is substituted for z, and then $\lambda(x)$ is substituted for y. Eliminating

$\alpha, \beta, \gamma, \delta$ between these ten equations (13) and combining with (11a) which $\lambda, \mu, \phi_1, \phi_2, \phi_3, \phi_4, \phi_5$ must also satisfy, we obtain for these functions the following equations

$$(14)\quad \begin{cases} (a) & F[x, \lambda(x), \mu(x), \phi_1(x), \phi_2(x), \phi_3(x), \phi_4(x), \phi_5(x)] = 0 \\ (b) & Ry'^2 - Sy' + T = 0 \\ (c) & z' = p + qy' \\ (d) & p' = r + sy' \\ (e) & q' = s + ty' \\ (f) & X + r'R + S'(S - Ry') = 0 \\ (g) & Y + s'R + t'\ (S - Ry') = 0. \end{cases}$$

These seven equations are however not independent, for we obtain the same result if we differentiate (14a) with respect to x totally, or if we multiply (14g) by y' and add it to (14f). The equations (14) then constitute a set of six independent ordinary differential equations for the seven variables y, z, p, q, r, s, t as functions of x. If one of them, say $y = \lambda(x)$, be chosen at pleasure, the remainder will be given uniquely if we further assign their values for any constant value of x. We will call a set of solutions of (14) a *characteristic torsion strip*,* since the torsion (r, s, t) as well as the tangent plane (p, q) will be determined at each point (x, y, z) of the curve. Hence follows the

THEOREM: *Given any preassigned set of constant values of the solutions, $x_0, y_0, z_0, p_0, q_0, r_0, s_0, t_0$, that is, a fixed torsion element at a point x_0, y_0, z_0, we have one and only one characteristic torsion strip through this element, corresponding to any choice of a certain arbitrary function. The simple strip contained in this characteristic torsion strip will be a characteristic strip, and the curve contained in this will, in turn, be a characteristic curve.* But more than one characteristic strip may be arranged along a given characteristic curve, and more than one characteristic torsion strip along each characteristic strip, although, *in general*, the characteristic curves, strips and torsion strips will be paired in a one-to-one manner.

For the linear equation

$$(15)\qquad A(x, y)r + B(x, y)s + C(x, y)t = D(x, y, z, p, q)$$

* See, *e. g.* Goursat, *l. c. 2 ordre*, I, p. 173, etc.

the equation (11) becomes

$$A(x, y)\left(\frac{dy}{dx}\right)^2 - B(x, y)\frac{dy}{dx} + C(x, y) = 0 \tag{11b}$$

which does not contain z, p, q, r, s, t. This case is then especially simple, though simplifications also arise in more general cases. Here it is not necessary to substitute for z, p, q, r, s, t, as in general. There are two one parameter families of solutions of (11b) of the form $y = \lambda(x)$. Any strip in space whose projection on the (x, y) plane is one of these curves, and which lies on an integral surface, is a characteristic strip. It is easy to show that the equations (14b), (14c), (14d), (14e) can be solved separately for z, p, q in this case. For having found $y = \lambda(x)$, a solution of (11b), it results from an algebraic calculation that we may replace the first five of equations (14) by the equations (15), (11b), and the equations

$$\begin{cases} z' = p + qa \\ p' = -\dfrac{C}{a}q' + \dfrac{D}{2A} \end{cases} \qquad \left(a = \frac{B \pm \sqrt{B^2 - 4AC}}{2A}\right)$$

and these equations contain neither r nor s. If $z = \mu(x)$ be taken at pleasure, these two equations can be solved to find $p = \phi_1(x)$, $q = \phi_2(x)$, involving, in general, one arbitrary constant. *Hence along any curve whose projection is one of the two parameter families of solutions of* (11b) *there exist, in general, a one parameter family of characteristic strips of the equation* (15).

From this point on, the development of the theory would follow exactly the ordinary course. We should next prove the two theorems:

(1) *Every integral surface of* (1) *containing a given torsion element* $(x_0, y_0, z_0, p_0, q_0, r_0, s_0, t_0)$ *and a certain characteristic curve on which the given torsion element lies, also contains the whole characteristic torsion strip determined by the given torsion element and the given characteristic curve, provided that* $S^2 - 4RT \neq 0$ *along the given characteristic curve.* Or, any two integral surfaces through the same characteristic curve, having double contact at one point of that curve, have double contact along its entire length, provided that $S^2 - 4RT \neq 0$ along the curve, which special case is left for later investigation.

(2) *A characteristic strip actually does not determine uniquely the integral surface on which it lies: — on the contrary, there are an infinite number of integral surfaces through this same strip; and in fact* (provided $S^2 - 4RT \not\equiv 0$ along the curves used) *if two characteristic curves belonging to different families pass through the same torsion element, one and only one integral surface exists which passes through the two curves.*

The proof of these two theorems,* as well as the remainder of the ordinary theory, we will pass over in silence. The statement of these theorems was, however, essential to establish our point of view of the characteristics, *as those curves which, together with their appended tangent planes, actually do not determine uniquely the integral surface on which they lie.*

It results from a further development that the theory of characteristics is valuable in the actual solution of problems, and whole classes of equations can be treated by methods based on it, so as eventually to effect their complete solutions.†

But it is clear that we cannot proceed to reduce the solution of the partial differential equation of the second order (1) to the solution of a set of ordinary differential equations, as we did in Part I for a partial differential equation of the first order. For the equations (14) comprise only six independent equations for the seven variables y, z, p, q, r, s, t as functions of x, and these are all the relations which can be obtained.

4. The Integral Strips. Let us now seek, as in Part I, §4, all those strips in space for which the Cauchy-Kowalewski theorem may fail in the above sense. Such strips we will call integral strips. It is evident that the characteristic strips will be included among the integral strips, and that an integral strip which is not a characteristic strip does not lie on any integral surface.

Since we do not assume that the strip

$$(16) \qquad \begin{cases} y = \lambda(x) & p = \phi_1(x) = \mu'(x) - \phi_2(x)\lambda'(x) \\ z = \mu(x) & q = \phi_2(x) \end{cases}$$

lies on an integral surface, it will be necessary to return to the substitutions for y, z, p, q, r, s, t, in Art. 2, in order to determine the substitutions which

* See Goursat, *l. c., 2 ordre*, II, pp. 184-5.

† See Goursat, *l. c., 2 ordre*, I and II.

we must make in (11) for y, z, p, q, r, s, t. These are given by the substitutions (6), (7), (9) in the following form:

$$(17)\quad \begin{cases} x = y_1 & = x \\ y = x_1 + \lambda(y_1) & = 0 + \lambda(x) \\ z = z & = \mu(x) \\ p = q_1 - \lambda' p_1 & = \mu'(x) - \phi_2(x)\,\lambda'(x) \\ q = p_1 & = \phi_2(x) \\ r = r_1\lambda'^2 - 2s_1\lambda' + t_1 - p_1\lambda'' & = \chi(x)\lambda'^2 - 2\phi_2'(x)\lambda'(x) + \mu''(x) - \phi_2(x)\lambda''(x) \\ s = s_1 - r_1\lambda' & = \phi_2'(x) - \chi(x)\,\lambda'(x) \\ t = r_1 & = \chi(x) \\ F(x, y, z, p, q, r, s, t) & = 0 \end{cases}$$

since we were to substitute $x_1 = 0$, $y_1 = y_1$, $z = \mu(y_1)$, $p_1 = \phi_2(y_1)$, $q_1 = \mu'(y_1)$, $r_1 = \chi(y_1)$, $s_1 = \phi_2'(y_1)$, $t_1 = \mu''(y_1)$; where $\chi(y_1)$ is determined by the condition that (1) be satisfied in the transformed variables.

The conditions (17) will be satisfied by the strip (16) if

$$(18)\quad \begin{cases} (a) & F(x, \lambda(x), \mu(x), \phi_1(x), \phi_2(x), r, s, t,) = 0. \\ (b) & \dfrac{d\mu(x)}{dx} = \phi_1(x) + \phi_2(x)\dfrac{d\lambda(x)}{dx} \\ (c) & \dfrac{d\phi_1(x)}{dx} = r + s\dfrac{d\lambda(x)}{dx} \\ (d) & \dfrac{d\phi_2(x)}{dx} = s + t\dfrac{d\lambda(x)}{dx} \end{cases}$$

which are the only conditions upon the functions of x which are to be substituted in (11) for y, z, p, q, r, s, t. In general r, s, and t may be eliminated between (11) and (18) so as to obtain an equation

$$(19)\qquad M\left(x, y, z, p, q, \frac{dy}{dx}, \frac{dp}{dx}, \frac{dq}{dx}\right) = 0.$$

This equation and (18b) will be the only equations which the strip (5) must satisfy in order to be an integral strip and we will therefore, by analogy to Part I, call (19) the *Monge equation* of the equation (1).

In general any two of the functions $\lambda(x)$, $\mu(x)$, $\phi_1(x)$, $\phi_2(x)$, may be

chosen at pleasure, and the other two found from (18b) and (19). Thus if $\lambda(x)$ and $\mu(x)$ are given, (19) reduces, in general, by means of (18b), to a single ordinary differential equation for $\phi_1(x)$, say. Then $\phi_2(x)$ is determined from $\phi_1(x)$ by (18b). *Hence there will, in general, be a one parameter family of integral strips through any curve in space*; but this may not be true in a particular example on account of the special form which the equations (18) may assume.

For instance, for the linear equation (15) the equation (11) becomes (11b), and it is not necessary to substitute for z, p, q, r, s, t, since only x and y occur in the equation. Having obtained, as above, the two one parameter families of curves in the XOY plane as solutions of (11b), it is clear that any strip of the form (16) will be an integral strip, if and only if $y=\lambda(x)$ is among these curves. *Hence any strip whatever through any curve in space whose projection is a solution of* (11b) *will be an integral strip of the equation* (15); *but no integral strip passes through any other curve in space.*

Returning to the general case of the equation (1), we obtain directly from the theorems of §3 the following

THEOREM: *No integral strip determines uniquely an integral surface, there being more than one integral surface through every characteristic strip, and none at all through any other integral strip.* For, by derivation, the characteristic strips are all those integral strips which lie on integral surfaces; and hence no other integral strip can possibly lie on any integral surface.

Let us then consider any strip S of the form (16), which is not an integral strip. It is clear from the Cauchy-Kowalewski theorem, and from §2, that such a non-integral strip will determine uniquely an analytic integral surface through itself, for any portion of S for which $F(x, y, z, p, q, r, s, t)$ is analytic, and for which (11) does not hold at any point $x_0, y_0, z_0, p_0, q_0, r_0, s_0, t_0$ of the interval, where r_0, s_0, t_0 are determined from (16) by (17) or (18). As in Part I, §4, we see that (11) cannot hold except at isolated points on S, since S is supposed not to be an integral strip. At the exceptional points where (11) is satisfied, S (with its strip) is necessarily tangent to some integral strip. For, by hypothesis, the given values of x, y, z, p, q determine at that point values of r_0, s_0, t_0 which, together with x_0, y_0, z_0, p_0, q_0, satisfy (11). Or, the values of $x_0, y_0, z_0, p_0, q_0, dy/dx]_0, dz/dx]_0, dp/dx]_0, dq/dx]_0$, obtained from S satisfy (19) and (18b); and it is clear that such a set of constant values, which satisfy (18b) and (19), determines in general one or more solutions of (18b) and (19), or of the equations (11) and (18) from which these are derived; *i. e.*, one

or more integral strips.* Any one of these integral strips is then tangent to S at the point x_0, y_0, z_0, in the sense that each of the torsion elements through the surface element x_0, y_0, z_0, p_0, q_0, in which the strip S lies also contains the integral strip in question.† For if the values of $\lambda'(x)$, $\mu'(x)$, $\phi'_2(x)$ coincide at x_0, y_0, z_0, p_0, q_0, then it appears from (17) that the values of r and s will coincide at that point, for *any arbitrary choice* of t (not necessarily satisfying (1)).

Excluding these isolated points on S where S is tangent to some integral strip, *i. e.* where $x_0, y_0, z_0, p_0, q_0, r_0, s_0, t_0$, determined from (16) by (17) or (18) satisfy (11), any remaining portion of S uniquely determines an analytic integral surface through itself, provided F is analytic at every point of this portion of S. As in Part I, §4, *if S lies wholly on an analytic integral surface, it determines that whole surface uniquely, irrespective of the above peculiar points.* And the only other cases which can arise, aside from a mixture of the three, are those in which (1) *S uniquely determines a single integral surface, which is analytic except at the excluded points*; or where (2) *the portions of S lying between excluded points determine pieces of different analytic integral surfaces.* It is clear, for example, that if two analytic integral surfaces be given which have a common characteristic strip (§3), then a non-integral analytic strip S may be drawn of which one portion lies on one of the given integral surfaces, while another portion lies on the other surface; the strip S being tangent to the given characteristic strip at the point where S crosses from one surface to the other. Such a non-integral strip evidently does not uniquely determine a single analytic integral surface along its whole extent, though each separate portion does uniquely determine an integral surface through itself. But it does determine uniquely an integral surface which is continuous and has a continuously turning tangent plane at the excluded point, and which is elsewhere analytic in the neighborhood of the strip, provided F is analytic along S in the interval considered. We may then state the

THEOREM: *Any strip S in space, of the form* (16), *which is not an integral strip, determines uniquely an integral surface which is analytic and which contains the given strip, for any portion of S in which no point lies at which S*

*Among these there will be, in general, one or more characteristic strips. Similarly in Part I, §4, there will be among the analogous integral curves in general *one and only one characteristic curve;* and the points to be excluded can be said with equal propriety to be those in which the curve in question (Part I) is tangent to a *characteristic curve*.

†The similar definition of tangency of two *curves* might have been used in Part I, §4: two curves are tangent at any point when each of the surface elements through that point, in which one of the curves lies, also contains the other curve.

is tangent to an integral strip, or at which $F(x, y, z, p, q, r, s, t,)$ *is non-analytic: the point* $x_0, y_0, z_0, p_0, q_0, r_0, s_0, t_0$ *being found from* (16) *by means of* (17) *or* (18). *But in any case any portion of* S *in which* F *is analytic at every point, uniquely determines an integral surface which is in general analytic, and which is at least continuous at the points where* S *is tangent to an integral curve; and the integral surface so determined may be analytic even at such points.*

It is clear that the above work, and in particular this last theorem, holds, with a suitable change of notation, for a strip of the form

$$\begin{cases} x = x_0 & p = \psi(y) \\ z = \mu(y) & q = \phi'(y) \end{cases}$$

which is of the type (3) for which the Cauchy-Kowalewski theorem was stated in §1.

For the linear equation (15) we have seen that the integral strips were all those strips in space which project into any curve of the two one parameter families of solutions of (11*b*). And we saw that, among these, the characteristic strips constitute in general a one parameter family along any space curve which projects into a solution of (11*b*). Any strip S in space, along a curve K, will be tangent (as a strip) to an integral strip, if and only if K is tangent to a curve which projects into a solution of (11*b*). In particular the strip S will certainly not be tangent to an integral strip if the projection of K on the XOY plane is not tangent to a solution of (11*b*). Hence we obtain for the linear equation (15) the following

THEOREM: *Given a curve* C *which projects into a solution of* (11*b*), *there is in general a one parameter family of characteristic strips through* C, *through each of which passes more than one integral surface of* (15). *Aside from these characteristic strips, no strip through* C *lies on any integral surface whatever, since all strips through* C *are integral strips. Any strip* S *in space along a curve* K, *which does not project into a solution of* (11*b*), *uniquely determines an analytic integral surface, for any portion of* S *in which no point lies at which* F *is non-analytic, or at which* K *is tangent to a curve* C *which projects into a solution of* (11*b*).

On account of the importance of the linear equation (15), and on account of the special rôle which the solutions of (11*b*) play in the theory, the two one parameter families of solutions of (11*b*) are themselves often called the "characteristics" of (15).

5. Relation to the Calculus of Variations.* As in the case of equations of the first order, for equations of the second order also, we find an intimate relation between the theory of characteristics and the Calculus of Variations.

Given the problem to render the integral

$$(1) \qquad I = \int z' d\tau$$

a minimum, under the auxiliary conditions

$$(2) \qquad F(x, y, z, p, q, r, s, t) = 0$$
$$(3) \qquad z' = px' + qy'$$
$$(4) \qquad p' = rx' + sy'$$
$$(5) \qquad q' = sx' + ty'$$

where $z' = \partial z/\partial t$, etc., τ being a parameter, while $x, y, z, p, q, r, s, t, F$ have the same meaning as in the last article. We transform the given problem again into the problem, to render the following integral a minimum, under no auxiliary conditions:

$$I' = \int \left[z' + \lambda(z' - px' - qy') + \mu(p' - rx' - sy') + \nu(q' - sx' - ty') + \pi F \right] d\tau$$

where λ, μ, ν, π represent undetermined functions of τ. We have as necessary conditions for the solution of the problem

$$(6) \begin{cases} (a) & \dfrac{d}{d\tau}(-p\lambda - \mu r - \nu s) = \pi \dfrac{\partial F}{\partial x} \\ (b) & \dfrac{d}{d\tau}(-q\lambda - \mu s - \nu t) = \pi \dfrac{\partial F}{\partial y} \\ (c) & \dfrac{d}{d\tau}(\lambda) = \pi \dfrac{\partial F}{\partial z} \\ (d) & \dfrac{d}{d\tau}(\mu) = -\lambda x' + \pi \dfrac{\partial F}{\partial p} = -\lambda x' + \pi P \\ (e) & \dfrac{d}{d\tau}(\nu) = -\lambda y' + \pi \dfrac{\partial F}{\partial q} = -\lambda y' + \pi Q \\ (f) & -\mu x' + \pi R = 0 \\ (g) & -(\mu y' + \nu x') + \pi S = 0 \\ (h) & -\nu y' + \pi T = 0 \end{cases}$$

* For the statements made in this article see Kneser, *Lehrbuch der Variationsrechnung.*

From (6c), $\frac{\lambda'}{\pi} = F_z$; from (6f), $\mu = \frac{\pi R}{x'}$; and from (6h), $\nu = \frac{\pi T}{y'}$.

Hence (6g) becomes

$$\frac{\pi R}{x'} y' + \frac{\pi T}{x'} y' - \pi S = 0,$$

or

$$R\left(\frac{dy}{dx}\right)^2 - S\frac{dy}{dx} + T = 0, \tag{7}$$

which coincides with equation (11) of the preceding article, the fundamental characteristic equation. Moreover, (6a) becomes on expansion

$$p\lambda' + p'\lambda + \mu r' + \mu' r + \nu s' + \nu' s + \pi F_x = 0. \tag{8}$$

Substituting from (6f) and (6h), and also from (6b), (6d), and (6e) we have

$$p\pi F_z + p'\lambda + r'\frac{\pi R}{x'} + r(\pi P - \lambda x') + s'\frac{\pi T}{y'} + s(\pi Q - \lambda y') + \pi F_x = 0,$$

or

$$\pi\left[pF_z + \frac{r'R}{x'} + Pr + \frac{Ts'}{y'} + Qs + F_x\right] + \lambda\left[p' - rx' - sy'\right] = 0.$$

But $p' - rx - sy' = 0$ by (4); and since $X = F_x + F_z p + Qs + P\gamma$, we have

$$X + R\frac{dr}{dx} + \frac{\frac{ds}{dx}T}{\frac{dy}{dx}} = 0. \tag{9}$$

Likewise from (6b) we obtain

$$Y + R\frac{ds}{dx} + \frac{\frac{dt}{dx}T}{\frac{dy}{dx}} = 0. \tag{10}$$

It only remains to show that (10) can be produced from (9), (7), and (2), as we have already done in § 3. We then have, for the determination of y, z, p, q, r, s, t, as functions of x which shall render the integral I' a minimum (or I under the conditions (2), (3), (4), (5)), the necessary conditions (7), (9) [or (10)], and (2), (3), (4), (5), which latter also result from (6) on proper elimination; and these six exhaust the first necessary conditions for the solution of the given problem. But these equations (7), (9), (2), (3), (4), (5) are exactly the equations (14) of § 3 for the characteristic torsion strips. In other words, *the solutions in y, z, p, q, r, s, t, as functions of*

x, of the equations (14), *are precisely the only possible solutions of the given problem of the Calculus of Variations; or, the only possible solutions of the given problem of the Calculus of Variations are given by the characteristic torsion strips of the equation* (2), *regarded as a differential equation.*

We might now easily go on to set up, as we have done in Part I, § 6 for equations of the first order, the equations of the common characteristics of two partial differential equations of the second order, and the condition that they be in involution, regarding the common characteristics as the common possible solutions of two problems in the Calculus of Variations, which reduce to a single problem of the type given above with one additional auxiliary condition of the type (2). Finally, we might seek the characteristics for two differential equations of the first order in two dependent variables, and so on. These problems will, however, offer no essential difficulty to the reader, and we will not enter into a discussion of them here; the main point of the existence of a connection between the theory of characteristics and the Calculus of Variations already having been demonstrated.

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FEBRUARY, 1903.

ERRATUM.

Page 139, line 7: instead of $q = \psi'(y)$, read $q = \phi'(y)$.

ON THE UNIFORMITY OF THE CONVERGENCE OF CERTAIN ABSOLUTELY CONVERGENT SERIES

BY MAXIME BÔCHER

IF the series

(1) $$u_1(x) + u_2(x) + \cdots$$

is absolutely and uniformly convergent for the values of x in a certain interval, and if we rearrange the terms, will the resulting series necessarily be uniformly convergent? This question must be answered in the negative as the following example shows:

(2) $$x^2 - x^2 + \frac{x^2}{1+x^2} - \frac{x^2}{1+x^2} + \frac{x^2}{(1+x^2)^2} - \frac{x^2}{(1+x^2)^2} + \cdots.$$

Here $$S_{2n} = 0, \qquad S_{2n+1} = \frac{x^2}{(1+x^2)^n}.$$

It is easily seen that S_{2n+1}, which reaches its maximum when $x = \pm 1/\sqrt{n-1}$, is always less than $1/n$. Accordingly (2) converges uniformly to the value zero. It is also readily shown to be absolutely convergent.

Now form the series

$$x^2 - x^2 + \frac{x^2}{1+x^2} + \frac{x^2}{(1+x^2)^2} - \frac{x^2}{1+x^2} + \frac{x^2}{(1+x^2)^3} + \frac{x^2}{(1+x^2)^4} - \frac{x^2}{(1+x^2)^3} + \cdots$$

from (2) by rearranging the terms. This series, of course, also converges to the value zero, but it converges non-uniformly in any interval including the point $x = 0$. For

$$S_{3n-1} = \frac{x^2}{(1+x^2)^n}\left[1 + \frac{1}{1+x^2} + \cdots + \frac{1}{(1+x^2)^{n-2}}\right] = \frac{(1+x^2)^{n-1} - 1}{(1+x^2)^{2n-2}},$$

and when x has either of the real values

$$\pm\sqrt{\sqrt[n-1]{2} - 1}$$

we have $S_{3n-1} = 1/4$.

It is, however, easy to prove the following theorem:

If the series

$$|u_1(x)| + |u_2(x)| + \cdots$$

is uniformly convergent throughout the interval in question, (1) *will be absolutely and uniformly convergent, and will remain so no matter how the order of the terms is changed.*

Any series which can be proved uniformly convergent by Weierstrass's test (*cf.* Weierstrass, *Werke*, Vol. II, p. 202) will therefore remain uniformly convergent when its terms are rearranged.

The facts here referred to would seem to be of especial interest in relation to the subject of absolutely convergent multiple series.

In conclusion I will mention the following theorem which I had occasion to prove and to use in a slightly different form a few years ago (*cf. Bull. Amer. Math. Soc.* May, 1898, p. 368):

If throughout the interval $a \leqq x \leqq b$ the functions $u_i(x)$ are continuous and nowhere negative, and if the function represented by the series (1) *is continuous throughout the interval $a \leqq x \leqq b$, then* (1) *converges uniformly throughout this interval.*

HARVARD UNIVERSITY, CAMBRIDGE, MASS.
JANUARY, 1903.

THE INTEGRAL AS THE LIMIT OF A SUM, AND A THEOREM OF DUHAMEL'S

By William F. Osgood

1. Introduction. A satisfactory treatment of the integral as the limit of a sum is to be found in any standard text-book on the calculus. If $f(x)$ is a real function of the real variable x, continuous throughout the interval $a \leqq x \leqq b$, then the sum

$$f(x_1')\Delta x_1 + f(x_2')\Delta x_2 + \cdots + f(x_n')\Delta x_n,$$

where $\Delta x_i = x_i - x_{i-1}$, $x_0 = a$, $x_n = b$, and $x_{i-1} \leqq x_i' \leqq x_i$, converges toward a limit when n becomes infinite, the longest Δx_i converging at the same time toward zero. Furthermore this limit, the definite integral of $f(x)$ taken between the limits a and b:

$$\int_a^b f(x)dx,$$

may be computed in terms of the indefinite integral

$$F(x) = \int f(x)dx$$

by means of the theorem that

$$\int_a^b f(x)dx = F(b) - F(a).$$

This and the corresponding theorem for multiple integrals, I say, are treated satisfactorily in standard treatises on the calculus.* The treatment of the application of these theorems to problems in physics and mechanics, both in courses and in text-books on the calculus, is however far from satisfactory, the attitude of the mathematician often being that these applications are without value for mathematics, while for the physicist any reasoning is good enough which in the long run leads to the right formula. To throw rigor to the winds as soon as the hypnotic influence of the environment of a course in

*For an excellent treatment of double and multiple integrals *cf.* Goursat, *Cours d'analyse mathématique*, Paris, 1902, ch. 6.

pure mathematics ceases is to regard rigor as a frill or as a luxury, as something having at most aesthetic reasons for existence,—not as a habit of thought, of practical value in research work. On the other hand the method to which this paper is devoted offers a valuable means of training students to appreciate the meaning of the integral calculus, and moreover a means which is alike valuable to students of applied and to students of pure mathematics. It quickens interest and develops power.

The method appears to be due to Duhamel* and it will be referred to in this paper as Duhamel's Theorem. It is contained in the principle (*cf.* §2 below) that in the limit of a sum of infinitesimals, any infinitesimal may be replaced by one that differs from it by one of higher order. It was introduced into calculus teaching in this country largely through Professor Byerly's, and Professor B. O. Peirce's text-books† and teaching.

Objections have been raised by Mansion to the ordinary formulation of the theorem.‡ His criticism is chiefly destructive, for while he points out how the theorem may be interpreted so as to be correct, the formulation to which he is led is, as he himself observes, of no use in practice, and his conclusion is that for this reason the only value which the theorem retains is for heuristic purposes,—as a "principe d'invention." In fact, what Mansion has done in this paper is similar to what Seidl§ did in pointing out that a convergent series of continuous functions does not necessarily define a continuous function,‖ and in finding out at what point Cauchy's proof of the contrary theorem breaks down. But just as Seidl failed to construct from his observations on series the principle of uniform convergence, and to introduce that principle as a means of investigation,—it was Weirstrass¶ who some seven years earlier, in two papers not published at the time, had taken this step: later, through his university lectures, the method became generally known,—so Mansion fails to extract from Duhamel's principle as originally stated a theorem formulated

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* Mansion (*cf.* below) refers it to Duhamel, who gives it in his *Eléments de calcul infinitésimal*, Paris, 1856, p. 35, and applies it to numerous problems in geometry and mechanics.

†Byerly, *Differential Calculus*, Boston, 1882, ch. x, Infinitesimals; *Problems in Differential Calculus*, Boston, 1895, ch. x B. O. Peirce, *Newtonian Potential Function*, 1st ed., Boston, 1886.

‡ *Mathesis*, vol. 8 (1888), p. 149.

§ *Abh. der kgl. bayerischen Akad. der Wiss.* (math.-phys. Cl.), vol. 5 (1848), p. 381.

‖ This fact had already been noted by Abel in his memoir on the binomial series, *Journ. für Math.*, vol. 1 (1826), p. 316, in the case of the Fourier's series $\Sigma(-1)^{n-1}\sin nx/n$; and it had also been observed by Stokes almost simultaneously with Seidl.

¶ *Cf.* two papers of the years 1841 and 1842, *Werke*, vol. 1, p. 67 and p. 75.

so simply that it can easily be applied to the cases that arise in practice, and thus to rescue Duhamel's principle for the important place in the calculus which it was designed to occupy. To put Duhamel's principle on a firm foundation and to make clear its importance in the applications of the calculus is the object of the present paper.

2. Duhamel's Theorem. Duhamel gives the following

THEOREM. *Let $a_1 + a_2 + \cdots + a_n$ be a sum of positive infinitesimals* which approaches a limit when $n = \infty$. Let $\beta_1 + \beta_2 + \cdots + \beta_n$ be a second sum of positive infinitesimals which differ respectively from the infinitesimals of the first sum by infinitesimals of higher order; i. e. let*

$$\lim_{n=\infty} \frac{\beta_i}{a_i} = 1.$$

Then the second sum approaches a limit when $n = \infty$, and this limit is the same as that of the first sum:

$$\lim_{n=\infty} (\beta_1 + \beta_2 + \cdots + \beta_n) = \lim_{n=\infty} (a_1 + a_2 + \cdots + a_n).$$

To prove the theorem let

$$\frac{\beta_i}{a_i} = 1 + \epsilon_i, \qquad \beta_i = a_i + \epsilon_i a_i,$$

and let the numerical value of the largest ϵ_i be denoted by ϵ:

$$|\epsilon_i| \leqq \epsilon, \qquad i = 1, 2, \cdots n.$$

Then

$$\begin{aligned} a_1 - \epsilon a_1 &\leqq \beta_1 \leqq a_1 + \epsilon a_1 \\ a_2 - \epsilon a_2 &\leqq \beta_2 \leqq a_2 + \epsilon a_2 \\ &\cdots\cdots\cdots \\ a_n - \epsilon a_n &\leqq \beta_n \leqq a_n + \epsilon a_n \end{aligned}$$

Adding, we get

$$(1 - \epsilon)\sum_{i=1}^{n} a_i \leqq \sum_{i=1}^{n} \beta_i \leqq (1 + \epsilon)\sum_{i=1}^{n} a_i.$$

Since $\lim \epsilon = 0$ when $n = \infty$, each extreme of this double inequality approaches as its limit $\lim \Sigma a_n$, and hence $\Sigma \beta_n$ approaches this limit also, *q. e. d.*

* By an infinitesimal is meant a *variable* whose *limit* is zero.

3. Applications. (*a*) *Attraction of a Rod.* **Let it be required to find the attraction of a straight rod of variable density on a particle situated in the line of the rod.**

Here we start with the physical law of gravitation, namely that two particles of masses m and m' attract each other with a force f proportional to their masses and inversely proportional to the square of the distance r between them:

$$f = \kappa \frac{mm'}{r^2}.$$

Now the rod is not made up of particles, but is a continuous distribution of matter of density ρ. How can we apply the law of gravitation to it? As follows: Divide it up into n parts, and denote the length of the i-th segment $x_i - x_{i-1}$ by Δx_i. Let ρ_i', ρ_i'' denote respectively the minimum and the maximum densities in this segment. Then the mass ΔM_i of so much of the rod obviously lies between $\rho_i'\Delta x_i$ and $\rho_i''\Delta x_i$:

$$\rho_i'\Delta x_i \leqq \Delta M_i \leqq \rho_i''\Delta x_i,$$

the equality signs holding in the special case of constant density.

Denote the attraction of the rod which we seek to compute by A, the part of the attraction due to the i-th segment by ΔA_i, so that

$$A = \sum_{i=1}^{n} \Delta A_i.$$

Now we can approximate to ΔA_i as follows: If all the mass ΔM_i were concentrated in a point at the nearer end of the i-th segment, we should obtain an attraction $\kappa m \Delta M_i / x_{i-1}^2$ greater than the true attraction ΔA_i, and if we were to replace ΔM_i by the larger mass $\rho_i''\Delta x_i$, the attraction thus obtained would be still greater: hence

$$\Delta A_i < \kappa \frac{m\rho_i''\Delta x_i}{x_{i-1}^2}.$$

On the other hand, if we were to concentrate the smaller mass $\rho_i'\Delta x_i$ in a point at the further end of the i-th segment, we should obtain an attraction $\kappa m\rho_i'\Delta x_i / x_i^2$ smaller than ΔA_i. Hence

$$\kappa \frac{m\rho_i'\Delta x_i}{x_i^2} < \Delta A_i < \kappa \frac{m\rho_i''\Delta x_i}{x_{i-1}^2} \tag{1}$$

Here, m is at the point O: $x = 0$.

The expressions that form the extremes of this double inequality have the further property that they both differ from ΔA_i by a small percent of ΔA_i when Δx_i is small; in other words, either one differs from ΔA_i by an infinitesimal of a higher order; for, divide through, say, by the left hand expression:

$$1 < \frac{\Delta A_i}{\kappa \dfrac{m\rho_i' \Delta x_i}{x_i^2}} < \frac{\rho_i'' x_i^2}{\rho_i' x_{i-1}^2}.$$

When $n = \infty$ and all the Δx_i's approach 0 as their limit, the right hand member of this double inequality approaches 1, hence the middle ratio does, too, and we see that $\kappa m\rho_i' \Delta x_i / x_i^2$ differs from ΔA_i by an infinitesimal of a higher order.

We are now in a position to apply Duhamel's Theorem and thus to evaluate the attraction A by integration. Since

$$A = \sum_{i=1}^{n} \Delta A_i,$$

we may allow n to become infinite, each ΔA_i approaching 0 as its limit, and we shall still have

$$A = \lim_{n=\infty} \sum_{i=1}^{n} \Delta A_i.$$

But here, in dealing with the limit of the sum $\Sigma \Delta A_i$, we may replace ΔA_i by any more convenient infinitesimal which differs from it by an infinitesimal of higher order. If, now, we can find such an infinitesimal, of the form $f(x_i)\Delta x_i$, then the new limit:

$$\lim_{n=\infty} \sum_{i=1}^{n} f(x_i)\Delta x_i,$$

can be evaluated by integration, for the fundamental theorem of §1 will apply to it.

Such an infinitesimal is suggested by the form of the approximations in (1): it is:

$$\kappa \frac{m\rho_i \Delta x_i}{x_i^2},$$

where ρ_i is formed for the point $x = x_i$: the function $f(x)$ being

$$f(x) = \kappa \frac{m\rho}{x^2}.$$

For, dividing (1) through by it:

$$\frac{\rho_i'}{\rho_i} < \frac{\Delta A_i}{\kappa \dfrac{m\rho_i \Delta x_i}{x_i^2}} < \frac{\rho_i'' x_i^2}{\rho_i x_{i-1}^2},$$

and the limit of each extreme is 1.

Hence by Duhamel's Theorem and the theorem in §1 we infer that

$$A = \lim_{n=\infty} \sum_{i=1}^{n} \Delta A_i = \lim_{n=\infty} \sum_{i=1}^{n} \kappa \frac{m\rho_i \Delta x_i}{x_i^2} = \int_a^b \kappa \frac{m\rho}{x^2}\, dx,$$

and the attraction of the rod is formulated as a definite integral. If, in particular, ρ is a constant, then

$$A = \kappa m\rho \int_a^b \frac{dx}{x^2} = \kappa m\rho \left(\frac{1}{a} - \frac{1}{b}\right).$$

The mass M of the rod is here $\rho(b - a)$, and thus

$$A = \kappa \frac{mM}{ab}.$$

The rod attracts, therefore, like a particle of the same mass concentrated, not at the centre of gravity of the rod, but at a distance from O which is a mean proportional between a and b.

(*b*) *Attraction of Any Material Body.* Let us now determine the attraction of an arbitrary material body on a particle. We begin by analyzing the method.

Let a system of Cartesian coordinates be assumed, and let the coordinates of the particle m be denoted by (a, b, c). We wish to find the components X, Y, Z along these axes of the attraction which the body exerts on m. Begin with the component X. Divide the volume V occupied by the body into n small pieces $\Delta V_1 \ldots \Delta V_n$ and denote the component attraction of the mass ΔM_i contained in the volume ΔV_i by ΔX_i. Then

$$X = \sum_{i=1}^{n} \Delta X_i.$$

The method is now this. If we allow n to become infinite and divide V up in any manner such that the maximum diameter of each ΔV_i approaches 0 as its limit, then, of course,

$$X = \lim_{n=\infty} \sum_{i=1}^{n} \Delta X_i. \tag{2}$$

This limit, as it stands, we do not know how to compute; but we do know how to compute the limit of a sum of the following form:

$$\lim_{n=\infty} \sum_{i=1}^{n} f(x_i, y_i, z_i) \Delta V_i, \tag{3}$$

where $f(x, y, z)$ denotes a function continuous throughout V. For, by the fundamental theorem of the integral calculus* referred to in §1, this limit exists and is the triple integral of $f(x, y, z)$ extended throughout V:

$$\iiint_V f(x, y, z) dV.$$

It may be computed by means of three successive simple integrations; for example, by the formula,

$$\int_a^b dx \int_{y_0}^{y_1} dy \int_{z_0}^{z_1} f(x, y, z) dz.$$

If, then, we can succeed in throwing the limit (2) by means of Duhamel's Theorem into the form of the limit (3), then we can evaluate (2) by simple integration. This can, in fact, be done, and we turn now to the proof that it can be. Assume for the present that the body lies wholly to the right of the plane $x = 0$; *i. e.* that at no point of the body x is negative.

We shut ΔX_i in between two expressions differing each from ΔX_i by a small percent of ΔX_i. The form of these approximations will suggest the form of the function $f(x, y, z)$ to be employed. Let ρ_i'', r_i'', α_i'' denote respectively the maximum values of the density ρ in ΔV_i, the distance from (a, b, c) to a point of ΔV_i, and the angle α which a ray drawn from (a, b, c) to a point of ΔV_i makes with the positive axis of x; and let ρ_i', r_i', α_i' denote respectively the minimum values of these functions. Then it is clear that if a particle of mass ΔM_i were concentrated at the shortest distance r_i' from m to a point of ΔV_i, and were

* *Cf.*, for example, Goursat, ch. 7.

placed, not necessarily at a point of ΔV_i, but on a ray making the minimum angle a_i' with the x-axis, the component attraction

$$\kappa \frac{m \Delta M_i}{r_i'^2} \cos a_i'$$

would be greater than ΔX_i: and if ΔM_i were replaced by the larger value $\rho_i'' \Delta V_i$ — we have as before

$$\rho_i' \Delta V_i \leqq \Delta M_i \leqq \rho_i'' \Delta V_i,$$—

then the inequality would only be strengthened. In a similar manner we obtain an expression that is less than ΔX_i, and hence the double inequality

$$\kappa \frac{m \rho_i' \Delta V_i}{r_i''^2} \cos a_i'' < \Delta X_i < \kappa \frac{m \rho_i'' \Delta V_i}{r_i'^2} \cos a_i'. \tag{4}$$

The first and last members above suggest a form of the summand in (3) such as is desired, namely, the summand obtained by setting

$$f(x, y, z) = \kappa \frac{m \rho}{r^2} \cos a.$$

For, dividing (4) through by this function formed for any arbitrary point of ΔV_i and multiplied by ΔV_i, we get

$$\frac{\rho_i' r_i^2 \cos a_i''}{\rho_i r_i''^2 \cos a_i} < \frac{\Delta X_i}{\kappa \frac{m \rho_i}{r_i^2} \cos a_i} < \frac{\rho_i'' r_i^2 \cos a_i'}{\rho_i r_i'^2 \cos a_i}. \tag{5}$$

When ΔV_i shrinks down toward a point as its limit, each extreme of this double inequality approaches 1 as its limit, and hence the middle term must do so also.

Thus the conditions of Duhamel's Theorem, §2, are satisfied and we have

$$\begin{aligned} X &= \iiint_V \kappa \frac{m \rho \cos a}{r^2} \, dV \\ &= \kappa m \iiint_V \frac{\rho (x - a) dx dy dz}{[(x-a)^2 + (y-b)^2 + (z-c)^2]^{3/2}}, \end{aligned} \tag{6}$$

with similar formulas for Y and Z.

If the solid lies wholly to the left of the plane $x = 0$, the component attraction in the direction of the negative axis of x can be computed in precisely the same manner: and if the solid lies on both sides of the plane, then the

attraction can be computed for each piece separately and the results, affected by the proper algebraic signs, added. The final formula will be the same as the above, (6).

4. Critique of Duhamel's Theorem and its Application in §3.

The infinitesimals α_i, β_i that appear in Duhamel's Theorem depend on two variables, i and n, and the *hypothesis*

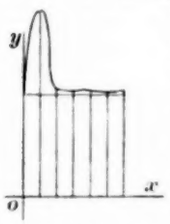
$$\lim_{n=\infty} \frac{\beta_i}{\alpha_i} = 1$$

tacitly assumes that i varies with n according to some specified law, or perhaps according to any law whatever; it is not explicit on this point. The *proof* tacitly assumes that $\lim \epsilon = 0$ when $n = \infty$.

An example will make clear questions which a critical examination of the law and its proof suggest.* Consider the curve

$$y = 1 + n^2 x e^{-n^2 x^2} \tag{7}$$

in the interval $0 \leqq x \leqq 1$. The graph of the curve is as indicated in the accompanying figure. For any specified value of x in the interval the ordinate of the curve approaches 1 as its limit when $n = \infty$; but the peak rises higher with increasing values of n and moves at the same time further and further to the left toward the axis of ordinates. At first sight it might appear as if the area under the curve necessarily approached as its limit the area under the limiting curve, $y = 1$; *i. e.* the area 1. But another factor comes in, namely, the ever increasing height of the peak, and so we see that it is not certain that the limit of the area under the curve is what it seemed to be. In fact, let us determine it: it is:



$$A_n = \int_0^1 y\,dx = \int_0^1 (1 + n^2 x e^{-n^2 x^2})\,dx$$

$$= (x - \tfrac{1}{2} e^{-n^2 x^2})\Big|_0^1 = 1 + \tfrac{1}{2}(1 - e^{-n^2}),$$

and $\lim A_n$ is seen to be $1\frac{1}{2}$ instead of 1.

Let us now divide the interval (0, 1) into n equal parts and through the points of division draw lines parallel to the axis of ordinates, thus dividing the

* This is the geometric form of an example given by Mansion, *l. c.*

area under the curve (7) into strips. As n increases, some of these strips will run up into the peak of the curve, so that their maximum altitude will increase indefinitely with n. In fact, let $\Delta x_i = 1/n$, $x_i = i\Delta x_i$. Then

$$y_i = 1 + nie^{-i^2}$$

and if i is held fast while n increases, y_i becomes infinite with n.

As the infinitesimal α_i of Duhamel's Theorem we will take the area of the i-th one of the above strips, namely,

$$\begin{aligned}\alpha_i &= \int_{x_{i-1}}^{x_i} (1 + n^2 x e^{-n^2x^2})\,dx \\ &= (x - \tfrac{1}{2}\, e^{-n^2x^2})\Big|_{x_{i-1}}^{x_i} \\ &= \Delta x_i + \tfrac{1}{2}\,(e^{-(i-1)^2} - e^{-i^2})\,;\end{aligned}$$

as β_i we will take the part of the area of this strip lying below the limiting curve $y = 1$, namely,

$$\beta_i = \Delta x_i.$$

Then if we fix our attention on the strip that contains a given point, as for example, the point $x = \lambda$, $y = 0$, and allow n to become infinite, i will become infinite at the same time and x_i, x_{i-1} will both approach λ as their limit. Hence we shall have

$$\lim_{n=\infty} \frac{\beta_i}{\alpha_i} = \lim_{n=\infty} \frac{1}{1 + \dfrac{n}{2}\left(e^{-n^2x_{i-1}^2} - e^{-n^2x_i^2}\right)} = 1$$

and the conditions of Duhamel's Theorem, as they might readily be understood, are fulfilled. But the theorem would then be false, for

$$\sum_{i=1}^{n} \alpha_i = \sum_{i=1}^{n} \Delta x_i + \sum_{i=1}^{n} \tfrac{1}{2}\left[(1 - e^{-1}) + (e^{-1} - e^{-4}) + \cdots + (e^{-(n-1)^2} - e^{-n^2})\right]$$

and

$$\lim_{n=\infty} \sum_{i=1}^{n} \alpha_i = 1\tfrac{1}{2}.$$

The explanation is simple. For the case in hand the maximum ϵ_i, taken numerically, namely ϵ, does not approach zero. Or again, it is possible to allow i so to vary with n that β_i/α_i does not approach 1; for if we hold i fast, and express Δx_i as $\dfrac{1}{n}$, then $\dfrac{\beta_i}{\alpha_i} = \left[1 + \dfrac{n}{2}\,(e^{-(i-1)^2} - e^{-i^2})\right]^{-1}$ and β_i/α_i approaches

0 as its limit when n becomes infinite. Here, the number of the strip remains constant and the strip moves further and further to the left.*

It is clear from this example that the formulation and the proof of the theorem leave something to be desired.†

The foregoing critique applies to Duhamel's Theorem itself. Let us now turn to its application in §3. First, the restriction that all the infinitesimals must be of like sign made a clumsy division into special cases necessary, the general result being identical with that obtained in each of the special cases.

A more serious difficulty is the following. Suppose the body extends up to the plane $x=0$. Then for a ΔV_i abutting on this plane $\alpha_i''=\frac{1}{2}\pi$ and $\cos \alpha_i''=0$. Hence it is not true in this case that the limit of the left hand member of (5) is unity, and the hypotheses of Duhamel's Theorem are not fulfilled. It would be possible, to be sure, to take the boundary of the solid distinct from the plane $x=0$, but near to it, and to allow the boundary subsequently to approach this plane as a limit. But to be obliged to take such precautions as these and go through with the consideration of a passage to a limit would detract very largely from the simplicity which is desirable in the case of so important a theorem. For the reasons here given we modify the statement of Duhamel's theorem in such a manner that it will apply to the cases that arise in practice and at the same time retain a satisfactory degree of simplicity.

5. Revised Formulation of Duhamel's Theorem. The important applications of Duhamel's Theorem in practice are to cases in which we wish to show that a given limit of a sum:

$$\lim_{n=\infty} \sum_{i=1}^{n} a_i$$

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* This example is closely related to the question of uniform convergence and the integration of a series term by term; *cf.* the author's geometric treatment of this subject, *Bulletin Amer. Math. Soc.*, ser. 2, vol. 3 (1896), p. 59.

† The formulation may be made precise and the theorem correct by requiring that $\lim \beta_i/\alpha_i = 1$, *no matter how* i *varies with* n, as n becomes infinite. For it can then be readily shown that $\lim_{n=\infty} \epsilon = 0$, and thus the proof of §2 will hold. But this formulation is of little use for practical purposes, since there is no simple sufficient condition which insures that the hypotheses of the theorem are fulfilled; and even if such a test were found, the theorem would still be inapplicable to simple cases that arise in practice, as will presently appear. The formulation given below (§5), on the other hand, is not open to this objection.

is equal to a definite integral,* either to

$$\lim_{n=\infty} \sum_{i=1}^{n} f(x_i)\Delta x_i = \int_a^b f(x)dx,$$

or to

$$\lim_{n=\infty} \sum_{i=1}^{n} f(x_i, y_i)\Delta S_i = \iint_S f(x,y)dS,$$

or to

$$\lim_{n=\infty} \sum_{i=1}^{n} f(x_i, y_i, z_i)\Delta V_i = \iiint_V f(x, y, z)dV.$$

We will restrict ourselves for simplicity to the first case.

Definition. Let ζ_i be defined by the equation

$$a_i = f(x_i)\Delta x_i + \zeta_i \Delta x_i.$$

Then the infinitesimal a_i is said to differ uniformly from the infinitesimal $f(x_i)\Delta x_i$ by an infinitesimal of higher order than Δx_i if, a positive quantity ϵ being chosen arbitrarily small, it is then possible to find a second positive constant δ such that, no matter how the interval $a \leqq x \leqq b$ be divided into n parts, we always have

$$|\zeta_i| < \epsilon,$$

provided merely that the division has proceeded far enough so that each $\Delta x_i < \delta$. Here a_i may be positive, negative, or zero.† In the case of a double or triple integral ζ_i will be defined by the equation

$$a_i = f(x_i, y_i)\Delta S_i + \zeta_i \Delta S_i \qquad \text{or} \qquad a_i = f(x_i, y_i, z_i)\Delta V_i + \zeta_i \Delta V_i,$$

where ΔS_i, ΔV_i denote respectively a piece of the surface or the volume into which the region of integration has been divided.

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*The notation here is not the same as in §2, the present a_i corresponding to the former β_i and the present $f(x_i)\Delta x_i$ to the former a_i.

†If one wishes to admit negative Δx_i's to the consideration, then the last relation must read

$$|\Delta x_i| < \delta.$$

THEOREM. *Let*

$$a_1 + a_2 + \cdots + a_n \qquad (A)$$

be a sum of infinitesimals and let a_i *differ uniformly by an infinitesimal of higher order than* Δx_i *from the summand* $f(x_i)\Delta x_i$ *of the definite integral*

$$\int_a^b f(x)dx \qquad (B)$$

of the function $f(x)$, *this function being continuous throughout the interval* $a \leqq x \leqq b$. *Then the sum* (A) *approaches a limit when* $n = \infty$, *and the value of this limit is the definite integral* (B):

$$\lim_{n=\infty} \sum_{i=1}^{n} a_i = \int_a^b f(x)dx.$$

Proof. Since

$$a_i = f(x_i)\Delta x_i + \zeta_i \Delta x_i$$

and

$$|\zeta_i| < \epsilon, \qquad \Delta x_i < \delta,$$

it follows that

$$\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} f(x_i)\Delta x_i + \sum_{i=1}^{n} \zeta_i \Delta x_i,$$

$$\left|\sum_{i=1}^{n} a_i - \sum_{i=1}^{n} f(x_i)\Delta x_i\right| \leqq \sum_{i=1}^{n} |\zeta_i| \Delta x_i < \sum_{i=1}^{n} \epsilon \Delta x_i = \epsilon(b-a), \qquad (8)$$

when $\Delta x_i < \delta$. Now the sum $\Sigma f(x_i)\Delta x_i$ converges toward the integral (B) when $n = \infty$, and so

$$\left|\sum_{i=1}^{n} f(x_i)\Delta x_i - \int_a^b f(x)dx\right| < \epsilon, \qquad (9)$$

when $\Delta x_i < \delta'$. If δ is chosen at least as small as δ', relations (8) and (9) will hold simultaneously as soon as $\Delta x_i < \delta$, $i = 1, \cdots n$. From (8) and (9) we infer that

$$\left|\sum_{i=1}^{n} a_i - \int_a^b f(x)dx\right| < \epsilon(b-a) + \epsilon,$$

when $\Delta x_i < \delta$, and this relation is coextensive with the assertion of the theorem we wished to prove.

Both theorem and proof apply without modification to multiple integrals.

6. Application of Duhamel's Revised Theorem to the Example of §3. The formulation of the problem down to the point of the double inequality (4) is the same as in §3. We now identify the infinitesimal ΔX_i with the a_i of §5 and set

$$f(x, y, z) = \kappa \frac{m\rho}{r^2} \cos a.$$

The functions ρ, r, $\cos a$ are continuous throughout the whole volume V inclusive of the boundary, and r never vanishes in this region. Hence ρ and $r^{-2}\cos a$ are both uniformly continuous throughout V, and the oscillation of each in ΔV_i can be made uniformly small by choosing all the ΔV_i's sufficiently small. If, then, η denote an arbitrarily small positive quantity, the oscillation in each ΔV_i will be less numerically than η for all divisions of V for which the maximum diameter of any ΔV_i is less than a suitably chosen positive constant δ. Moreover the maximum numerical value G of either function in V is finite:

$$|\rho| \leqq G, \qquad \left|\frac{\cos a}{r^2}\right| \leqq G.$$

Thus we shall have

$$\rho_i' = \rho_i + \theta_i^{(1)},$$

$$\frac{\cos a_i'}{r_i'^2} = \frac{\cos a_i}{r_i^2} + \theta_i^{(2)},$$

where ρ_i, r_i, $\cos a_i$ apply to an arbitrarily chosen point of ΔV_i and

$$|\theta_i^{(1)}| < \eta, \qquad |\theta_i^{(2)}| < \eta.$$

Hence

$$\kappa \frac{m\rho_i' \Delta V_i}{r_i'^2} \cos a_i' = \kappa \frac{m\rho_i \Delta V_i}{r_i^2} \cos a_i + \zeta_i' \Delta V_i,$$

where

$$\zeta_i' = \kappa m(r_i^{-2} \cos a_i \theta_i^{(1)} + \rho_i \theta_i^{(2)} + \theta_i^{(1)} \theta_i^{(2)}) \text{ and } |\zeta_i'| < \kappa m(2G + \eta)\eta = \epsilon.$$

A similar relation holds for the left hand member of (4). Hence if we subtract the expression

$$\kappa \frac{m\rho_i \Delta V_i}{r_i^2} \cos a_i$$

from each member of (4), the new extremes will each be less numerically than $\epsilon \Delta V_i$, and we get

$$|\Delta X_i - \kappa \frac{m\rho_i \Delta V_i}{r_i^2} \cos a_i| < \epsilon \Delta V_i.$$

This relation expresses precisely the fact that

$$\kappa \frac{m\rho_i \Delta V_i}{r_i^2} \cos a_i$$

differs uniformly from ΔX_i by an infinitesimal of higher order.

7. The Axioms of Physics. *Gravitation.* We have chosen for purposes of illustration a problem in gravitation. The method set forth applies in the same manner to problems in centres of gravity, moments of inertia, fluid pressures, kinetic energy of a rigid body (or, generally, of any continuous distribution of matter), work done on a rigid body by gravitational forces, etc. It is interesting to inquire in each case what the physical laws assumed are,—what is definition and what is proof.

Consider the problem of §3. Here, to begin with, the law of gravitation for two particles is assumed. But is that all? Have we, with the aid of this law alone, *computed* the attraction of the given body? By no means. We have assumed as further axioms

(a) that a resultant attraction for the whole body and for any piece of it, exists;

(b) that when the body is divided up in any way the resultant attraction due to the attractions of all the component pieces is equal to the resultant attraction of the whole body;

(c) that the intensity of the resultant attraction for a given piece is less than the force which a particle of the same mass as this piece would exert if placed at the minimum distance from the given particle to a point of the piece, and greater than if that particle were placed at the maximum distance;

(d) that the direction of the resultant attraction of any piece lies within any convex cone that can be drawn with the given particle as vertex, containing the piece in question.*

Moment of Inertia. An analogous set of axioms is employed in finding centres of gravity of continuous distributions of matter, and moments of inertia may be treated in the same way. Another point of view regarding moments of inertia is as follows. Consider, for example, the moment of inertia of a plane lamina of variable density about an axis perpendicular to its plane and

*This axiom regarding the direction of the resulting force is not the same in form as the one employed in §3; but it is evident that the inequality (4) of that paragraph can be deduced from it.

piercing the plane in the point O. Here we begin with the definition of the moment of inertia of a particle of mass m situated in the plane, about O as

$$I = mr^2,$$

where r denotes the distance of the particle from O, and we define the moment of inertia of a system of such particles as the sum of the moments of inertia of the individual particles. Now instead of considering the moment of inertia of the lamina about O as a fundamental concept and proceeding to compute its value as above suggested, we may regard this moment of inertia as a quantity to be defined and for the purpose of the definition lay down requirements suggested by the physical meaning of the moment of inertia of a particle or system of particles. These requirements we may state as follows: The moment of inertia of the lamina, and of any piece of it, shall be so defined,

(a) that when the lamina is divided up in any way, the sum of the moments of inertia of the component pieces shall equal the moment of inertia of the whole lamina:

(b) that the moment of inertia of a piece shall lie between the moments of inertia of a particle of the same mass when concentrated at the nearest point of the piece and at the farthest point respectively.

The question that now presents itself is whether such a definition is possible, and if so, how the moment of inertia may be computed. Let us see. Suppose that the definition is possible. Denote the moment of inertia of the whole lamina by I, that of one of the component parts into which it is divided by ΔI_i. Then, if ΔM_i, ΔS_i, ρ_i', ρ_i'', r_i', r_i'' denote respectively the mass, the area, the minimum and the maximum densities, and the minimum and the maximum distances from O to a point of ΔS_i, we shall have

$$I = \sum_{i=1}^{n} \Delta I_i = \lim_{n=\infty} \sum_{i=1}^{n} \Delta I_i,$$

$$\rho_i' \Delta S_i \leqq \Delta M_i \leqq \rho_i'' \Delta S_i,$$

$$\rho_i' \Delta S_i r_i'^2 < \Delta I_i < \rho_i'' \Delta S_i r_i''^2.$$

The extremes of the last inequality suggest the function

$$f(x, y) = \rho r^2$$

formed for an arbitrary point (x_i, y_i) of the region ΔS_i. Now ρ and r are both continuous functions of x, y throughout the whole region of the lamina; hence

they are uniformly continuous, and thus we are able to show, just as in §6, that ΔI_i differs uniformly from

$$\rho_i r_i^2 \Delta S_i$$

by an infinitesimal of higher order. All of the conditions of the revised form of Duhamel's Theorem, §5, are fulfilled regarding the infinitesimals $\Delta_i I$ and we see that I must have the value

$$I = \lim_{n=\infty} \sum_{i=1}^{n} \rho_i r_i^2 \Delta S_i = \int\int_S \rho r^2 dS.$$

This is a necessary condition for the existence of a moment of inertia which shall satisfy the given requirements. Conversely, it is sufficient, as is readily shown by the law of the mean. Hence we may define the moment of inertia of the lamina as

$$I = \int\int_S \rho r^2 dS,$$

and this shall be its definition.

Length of Arc of a Curve. For the elementary student the length of the arc of a curve is a fundamental concept. He is ready to accept the axiom (*i.e.* physical law) that the length of an arc of a plane convex curve, when the arc lies wholly on one side of the straight line which passes through its extremities, is less than the length of a broken line enveloping it and having the same extremities; while on the other hand, the length of the arc is greater than that of the chord. The ordinary formula for the length of an arc of a plane curve may then be proved as in §6. On the other hand, for the professional mathematician the length of the arc of a curve is a matter of definition, and the requirements on which the definition depends lead to considerations analogous to those above discussed in the case of the definition of the moment of inertia.

8. Conclusion. In closing, one word about the place of these theorems in elementary instruction in the calculus. It is advisable that the integral as the limit of a sum be introduced at an early stage and illustrated by numerous problems in the determination of volumes of geometric solids. The solid in question is sliced up and the actual slice is compared with an auxiliary slice whose volume is readily computed in the form

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$$I = \sum_{i=1}^{n} \Delta I_i = \lim_{n=\infty} \sum_{i=1}^{n} \Delta I_i,$$

$$\rho_i' \Delta S_i \leqq \Delta M_i \leqq \rho_i'' \Delta S_i,$$

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they are uniformly continuous, and thus we are able to show, just as in §6, that ΔI_i differs uniformly from

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$$f(x_i)\Delta x_i$$

and which is so chosen that the total volume of these auxiliary slices,

$$\sum_{i=1}^{n} f(x_i)\Delta x_i, \tag{10}$$

obviously converges toward the volume sought as a limit. But the limit of (10) can be determined by integration, §1. In many cases, as for example in finding the volume of a regular pyramid or of an anchor ring, a formal proof may readily be given, and it is desirable that some such proofs be given. But the observational faculties of the student should also be exercised and formal proofs, when clumsy, may well be waived in favor of greater stress on the intuitional side of the problem.

Proceeding now to the next stage, the treatment of physical problems, we find that intuition is less clear. The student may be willing to grant the formulation of the attraction of the rod in §2 as the definite integral there found for it, but his reasoning is more likely to be based on formal analogy, such as led him a year earlier to think that $\sin 2x$ is equal to $2 \sin x$, than on an estimate of the relative magnitudes of the infinitesimals with which he is dealing. It is here that Duhamel's Theorem in its original form (§2) gives him a rigorous principle by means of which to determine whether his processes are correct. I say a *rigorous* principle, for the principle meets the highest demands for rigor of which the student is capable at this stage. What is the state of affairs at present regarding the treatment of these physical problems? The infinitesimal is treated as an inconceivably small constant, and an infinitesimal which does not succeed in making itself *persona grata* is neglected. Shall we hesitate to replace this barbarous method by one right in principle and easy of application simply because the whole truth can not be told at this point? The systematic treatment of physical problems by the method of limits and Duhamel's Principle means a step in advance in calculus teaching, and it is possible and desirable for this generation of mathematicians to take this step, both in college courses and in the courses of our technical schools.

Finally the student of mathematical physics and of higher analysis, who has reached the stage at which questions of uniform convergence are treated, will find in the revised form of Duhamel's Theorem a method rigorous according to the highest standards of today.

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ON A GENERAL RELATION OF CONTINUED FRACTIONS

By Robert E. Moritz

Let us write with Dirichlet,*

$$(a_1, \cdots, a_n) = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots + \cfrac{1}{a_{n-1} + \cfrac{1}{a_n}}}}; \tag{1}$$

it has then been shown that

$$(a_1, \cdots, a_n)(a_n, \cdots, a_2) = (a_n, \cdots, a_1)(a_1, \cdots, a_{n-1}), \tag{2}$$

a relation which Moebius† has characterized as one of the most remarkable relations between continued fractions.

We propose to establish the general relation

$$\prod_{k=k}^{k=l-1}(a_k, \cdots, a_n) \prod_{n=n}^{n=m+1}(a_n, \cdots, a_l) = \prod_{n=n}^{n=m+1}(a_n, \cdots, a_k) \prod_{k=k}^{k=l-1}(a_k, \cdots, a_m), \tag{3}$$

$$k < l < m < n,$$

of which (2) is but a very special case. Incidentally we shall arrive at a simple rule for expressing a quotient of two continuants as a product of continued fractions.

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* Dirichlet's *Werke*, vol. 2, p. 141.

† Moebius, *Crelle's Journal*, vol. 6 (1830), p. 218.

Let the continuant

$$p_{r,t} = \begin{vmatrix} a_r & 1 & 0 & . & . & . & . \\ -1 & a_{r+1} & 1 & . & . & . & . \\ 0 & -1 & a_{r+2} & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & a_{t-2} & 1 & 0 \\ . & . & . & . & -1 & a_{t-1} & 1 \\ . & . & . & . & 0 & -1 & a_t \end{vmatrix}, \quad r \leqq t, \qquad (4)$$

be developed in terms of the minors of its first $s - r + 1$ rows or columns, and we get

$$p_{r,t} = p_{r,s}\, p_{s+1,t} + p_{r,s-1}\, p_{s+2,t}, \quad r \leqq s \leqq t. \qquad (5)$$

In order that this equation may not be meaningless when $s = t - 1$, or $s = t$, we must put

$$p_{t+1,t} = 1, \qquad p_{t+2,t} = 0, \qquad (6)$$

which is allowable since $p_{t+1,t}$ and $p_{t+2,t}$ have not yet been defined.

For $s = r$, and $s = t - 1$, we have

$$p_{r,t} = a_r p_{r+1,t} + p_{r+2,t}, \qquad r \leqq t, \qquad (7)$$

and

$$p_{r,t} = a_t p_{r,t-1} + p_{r,t-2}, \qquad r \leqq t, \qquad (8)$$

respectively.

By properly interchanging the rows and then the columns in (4) and multiplying the elements in each row and column by -1, we have also

$$p_{r,t} = p_{t,r}, \qquad r \leqq t. \qquad (9)$$

Equations (7) and (8) furnish immediately a well known relation between continuants and continued fractions: for, dividing (7) by $p_{r+1,t}$, we have

$$\frac{p_{r,t}}{p_{r+1,t}} = a_r + \cfrac{1}{\cfrac{p_{r+1,t}}{p_{r+2,t}}},$$

and in like manner

$$\frac{p_{r+1,t}}{p_{r+2,t}} = a_{r+1} + \cfrac{1}{\cfrac{p_{r+2,t}}{p_{r+3,t}}},$$

. . . . .

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\frac{p_{t-1,t}}{p_{t,t}} = a_{t-1} + \frac{1}{\dfrac{p_{t,t}}{p_{t+1,t}}},$$

$$\frac{p_{t,t}}{p_{t+1,t}} = a_t.$$

Consequently we have by successive substitution

$$\frac{p_{r,t}}{p_{r+1,t}} = (a_r, \cdots, a_t), \qquad r \leqq t, \tag{10}$$

and similarly we get from (8)

$$\frac{p_{r,t}}{p_{r,t-1}} = (a_t, \cdots, a_r), \qquad r \leqq t. \tag{11}$$

Since the reciprocal of a continued fraction is again a continued fraction, we may write

$$\frac{1}{(a_r, \cdots, a_t)} = (0, a_r, \cdots, a_t) = (a_r, \cdots, a_t)'. \tag{12}$$

We are now prepared to prove the following proposition:

Every quotient of two continuants, $p_{k,m}/p_{l,n}$, can be expressed in two different ways as a product of continued fractions of the type $(a_r, \ldots, a_s)$.

For the sake of simplicity we shall assume for the present that k, l, m, n are all different, furthermore that the quotient $p_{k,m}/p_{l,n}$ is so written that $k < m$, and $l < n$, an assumption which is permissible because of (9). Again, we may assume without loss of generality that $k < l$, for if $k > l$ we need only consider the reciprocal $p_{l,n}/p_{k,m}$ for which then the assumed condition holds. After this reciprocal quotient has been expressed as a product of continued fractions, we get the required expression for the original quotient by changing every primed factor $(a_r, \ldots, a_s)'$ into an unprimed factor $(a_r, \ldots, a_s)$, and *vice versa*.

Because of the foregoing considerations, the twenty-four possible cases reduce to the three following:

(1) $k < l < m < n$,
(2) $k < l < n < m$,
(3) $k < m < l < n$.

Case 1. $k < l < m < n$.

Here we have

$$\frac{p_{k,m}}{p_{l,n}} = \frac{p_{k,m}}{p_{k+1,m}} \frac{p_{k+1,m}}{p_{k+2,m}} \cdots \frac{p_{l-1,m}}{p_{l,m}} \frac{p_{l,m}}{p_{l,m+1}} \frac{p_{l,m+1}}{p_{l,m+2}} \cdots \frac{p_{l,n-1}}{p_{l,n}}.$$

Now equation (10) gives

$$\frac{p_{k,m}}{p_{k+1,m}} = (a_k, \cdots, a_m), \quad \frac{p_{k+1,m}}{p_{k+2,m}} = (a_{k+1}, \cdots, a_m), \text{ etc.},$$

and by (11) and (12)

$$\frac{p_{l,m}}{p_{l,m+1}} = \frac{1}{(a_{m+1}, \cdots, a_l)} = (a_{m+1}, \cdots a_l)', \text{ etc.};$$

hence

$$\begin{aligned} \frac{p_{k,m}}{p_{l,n}} = {} & (a_k, \cdots, a_m)(a_{k+1}, \cdots, a_m) \cdots (a_{l-1}, \cdots, a_m) \\ & \times (a_{m+1}, \cdots, a_l)'(a_{m+2}, \cdots, a_l)' \cdots (a_n, \cdots, a_l)', \end{aligned} \tag{13}$$

and similarly

$$\begin{aligned} \frac{p_{k,m}}{p_{l,n}} &= \frac{p_{k,m}}{p_{k,m+1}} \frac{p_{k,m+1}}{p_{k,m+2}} \cdots \frac{p_{k,n-1}}{p_{k,n}} \frac{p_{k,n}}{p_{k+1,n}} \frac{p_{k+1,n}}{p_{k+2,n}} \cdots \frac{p_{l-1,n}}{p_{l,n}} \\ &= (a_{m+1}, \cdots, a_k)'(a_{m+2}, \cdots, a_k)' \cdots (a_n, \cdots, a_k)' \\ &\quad \times (a_k, \cdots, a_n)\,(a_{k+1}, \cdots, a_n) \cdots (a_{l-1}, \cdots, a_n). \end{aligned} \tag{13'}$$

Case 2. $k < l < n < m$.

In like manner

$$\begin{aligned} \frac{p_{k,m}}{p_{l,n}} &= (a_k, \cdots, a_m)(a_{k+1}, \cdots, a_m) \cdots (a_{l-1}, \cdots, a_m) \\ &\quad \times (a_m, \cdots, a_l)(a_{m-1}, \cdots, a_l) \cdots (a_{n+1}, \cdots, a_l) \qquad (14) \\ &= (a_m, \cdots, a_k)(a_{m+1}, \cdots, a_k) \cdots (a_{n+1}, \cdots, a_k) \\ &\quad \times (a_k, \cdots, a_n)(a_{k+1}, \cdots, a_n) \cdots (a_{l-1}, \cdots, a_n). \qquad (14') \end{aligned}$$

An inspection of (13), (13′), (14), (14′) leads to the following *memoria technica* which enables us to write down immediately not only these products, but the products for any quotient of the form $p_{k,m}/p_{l,n}$.

1. *Write down the indices of the elements in the continuant of the numerator in a horizontal line, and under them write the indices of the elements*

in the continuant of the denominator, but in such a manner that each index shall stand under the like index in the first line. Let k and m represent respectively the first and last indices in the upper line, l and n respectively the first and last indices of the lower line. We then obtain the required products by using as factors the following continued fractions:

2. *Those fractions in which the elements, lying between a_k and a_l and including that one of these two which has the lesser index, form the initial elements and a_m (or a_n) the end elements.*

3. *Those fractions in which the elements, lying between a_m and a_n and including that one of these two which has the greater index, form the initial elements and a_l (or a_k) the end elements.*

4. *Prime all those fractions whose initial indices are found in the lower line but not in the upper.*

The equations (13), (13′), (14), (14′) remain true even if some of the indices l, k, m, n are equal. In that case we need only suppress such factors as lose their sense.

For instance, when $l = k$, (14) and (14′) go over into

$$k < n < m,$$

$$\frac{p_{k,m}}{p_{k,n}} = (a_m, \cdots, a_k)(a_{m-1}, \cdots, a_k) \cdots (a_{n+1}, \cdots, a_k), \qquad (15)$$

and for $n = m$, (13) or (13′) gives us

$$k < l < m,$$

$$\frac{p_{k,m}}{p_{l,m}} = (a_k, \cdots, a_m)(a_{k+1}, \cdots, a_m) \cdots (a_{l-1}, \cdots, a_m). \qquad (16)$$

Of special interest is the case in which the continuant of the denominator reduces to a single element, that is, when in (15) $k = n$, or in (16) $l = m$. Remembering that $p_{k,k} = a_k$ and $p_{m,m} = a_m$, we have

$$p_{k,m} = (a_m, \cdots, a_k)(a_{m-1}, \cdots, a_k) \cdots (a_{k+1}, a_k)\, a_k, \quad k < m, \quad (17)$$

$$p_{k,m} = (a_k, \cdots, a_m)(a_{k+1}, \cdots, a_m) \cdots (a_{m-1}, a_m)\, a_m, \quad k < m, \quad (18)$$

that is,

Every continuant can be expressed in two ways as a product of continued fractions.

From (17) and (18) it appears immediately that the denominator of any reduced continued fraction on the right equals the numerator of the continued

fraction next succeeding it, so that after reduction there remains only the numerator of the first fraction on the right in each equation. We have thus arrived at the well-known theorem:

The continued fractions $(a_m, \cdots, a_k)$ and $(a_k, \cdots, a_m)$ *have equal numerators*;

and to the following simple rule for computing a given continuant:

To find the value of $p_{k,m}$ *compute the numerator of either of the fractions* $(a_k, \cdots, a_m)$ or $(a_m, \cdots, a_k)$.

Case 3. $k < m < l < n$.

Formulae (17) and (18) enable us also to dispose of the remaining third case in the simplest way possible. For we need only write the product for $p_{k,m}$ and $p_{l,n}$ separately and then multiply the fractions of the first by the reciprocals of the factors of the second.

If we equate (13) and (13′) and remember that any primed factor equals the reciprocal of the corresponding unprimed factor we easily deduce

$$\begin{aligned}&(a_k, \cdots, a_n)(a_{k+1}, \cdots, a_n) \cdots (a_{l-1}, \cdots, a_n)\\ &\qquad \times (a_n, \cdots, a_l)(a_{n-1}, \cdots, a_l) \cdots (a_{m+1}, \cdots, a_l)\\ &= (a_n, \cdots, a_k)(a_{n-1}, \cdots, a_k) \cdots (a_{m+1}, \cdots, a_k)\\ &\qquad \times (a_k, \cdots, a_m)(a_{k+1}, \cdots, a_m) \cdots (a_{l-1}, \cdots, a_m), \qquad (19)\end{aligned}$$

or, as it may be otherwise written,

$$\prod_{k=k}^{k=l-1}(a_k, \cdots, a_n) \prod_{n=n}^{n=m+1}(a_n, \cdots, a_l) = \prod_{n=n}^{n=m+1}(a_n, \cdots, a_k) \prod_{k=k}^{k=l-1}(a_k, \cdots, a_m). \qquad (20)$$

Equations (14) and (14′) furnish essentially the same relation.

Moebius' relation

$$(a_1, \cdots, a_n)(a_n, \cdots, a_2) = (a_n, \cdots, a_1)(a_1, \cdots, a_{n-1})$$

is obtained from (20) by putting $k = 1$, $l = 2$, $m = n - 1$.

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NOTE ON THE EQUILATERAL HYPERBOLA

By John A. Van Groos

1. The equilateral hyperbola $x^2 - y^2 = 1$ can be represented by the equations:

$$x = \cosh t, \qquad y = \sinh t,$$

where the right branch of the hyperbola corresponds to the values of t from $-\infty$ to $+\infty$, and the left branch to the values from $-\infty + \pi i$ to $+\infty + \pi i$.

When $t = 0$ we get the point $(1, 0)$; when $t = 0 + \pi i$, the point $(-1, 0)$. The points $-t$ and $-t + \pi i$ are symmetrical to the point t with respect to the axes of X and Y respectively; the points t and $t \pm 2n\pi i$ are of course identical. If the pure imaginary part of t is not a multiple of π, the corresponding point (x, y) will be imaginary.

For real values of the parameter, t equals twice the area of a sector whose sides are the major axis and the radius vector to the point t.

2. *If the three vertices t_1, t_2, t_3, of a triangle lie on the curve, then its orthocentre (point of intersection of the altitudes) will also lie on the curve.*

This well known theorem is due to Brianchon and Poncelet,* and is easily proved as follows:

Let the three vertices be $t_1 = (x_1, y_1)$, $t_2 = (x_2, y_2)$, and $t_3 = (x_3, y_3)$: then the equation of the line $t_1 t_2$ is

$$\frac{y - y_1}{x - x_1} = \frac{y_1 - y_2}{x_1 - x_2} = \frac{\sinh t_1 - \sinh t_2}{\cosh t_1 - \cosh t_2} = \coth \tfrac{1}{2}(t_1 + t_2),$$

and the equation of the perpendicular to this line through t_3 is

$$\frac{y - y_3}{x - x_3} = -\tanh \tfrac{1}{2}(t_1 + t_2). \tag{1}$$

Let $t_0 = (x_0, y_0)$ be the point where this perpendicular cuts the hyperbola. Then we have

$$\coth \tfrac{1}{2}(t_0 + t_3) = -\tanh \tfrac{1}{2}(t_1 + t_2), \tag{2}$$

whence

$$t_0 \equiv -(t_1 + t_2 + t_3) + \pi i \pmod{2\pi i}. \tag{3}$$

*Gergonne's *Annales*, vol. 11, pp. 205-220, Jan. 1, 1821.

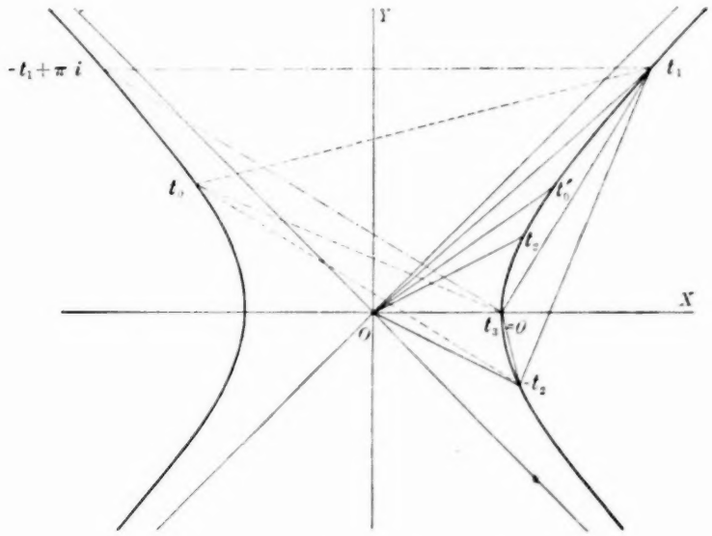
If we take a second altitude of the triangle, we find, in the same way, that this line cuts the hyperbola in the same point t_0.

3. From the relation (3) we deduce at once the theorem:

If 3^n points on an equilateral hyperbola be grouped in 3^{n-1} triangles in any way, they determine 3^{n-1} orthocentres. These orthocentres grouped in any way determine 3^{n-2} new orthocentres, and so on. The last three points determine a final point on the hyperbola. No matter how the points are grouped into triangles, the final point is always the same, its parameter being

$$t_0 \equiv -\sum_{\kappa=1}^{3^n} t_\kappa + \pi i \qquad (\mathrm{mod}\ 2\pi i). \tag{4}$$

4. The relation (3) enables us also to add geometrically any two sectors of an equilateral hyperbola.



First, to reduce any sector t_1Ot_2 (see figure) to an equivalent sector having the major axis (from O to $t_3 = 0$) as a side, find the point $-t_2$ symmetric to t_2 with respect to the X-axis, construct the orthocentre t_0 of the points $t_1, -t_2, t_3$, and find its symmetric point t_0' with respect to the Y-axis: then the sector under t_0' will be the sector required. For, by (3), $t_0 = -(t_1 - t_2 + 0) + \pi i$, and therefore $t_0' = -t_0 + \pi i = t_1 - t_2$.

Secondly, to add two sectors which have the major axis as a common side, as the sectors under t_1 and t_2 in the figure, construct the orthocentre of the points t_1, t_2, t_3, and its symmetric point with respect to the axis of Y; the sector under this last point will be equivalent to the sum of the sectors under t_1 and t_2.

5. If we take two vertices, t_1 and $t_2 = -t_1 + \pi i$, of our inscribed triangle on a line parallel to the X-axis, and let the third vertex fall at a vertex, $t_3 = 0$, of the hyperbola, we shall get for the orthocentre

$$t_0 = -(t_1 - t_1 + \pi i + t_3) + \pi i = -t_3 : \tag{5}$$

that is, the orthocentre will fall at $t_3 = 0$, and the triangle will be right angled.

*Hence, in an equilateral hyperbola any chord parallel to the principal axis subtends a right angle at either vertex of the curve.**

If now t_3 moves upward to t_1, the orthocentre t_0 will move downward, always keeping symmetric to t_3 with respect to the X-axis, by (5). Hence in the limit, when t_3 reaches t_1, and $t_3\,t_1$ becomes a tangent to the curve at t_1, the point t_0 will be symmetrical to t_1 with respect to the major axis.

Hence, to draw a tangent at any point t_1, find its symmetric points on the hyperbola with respect to the x and y axes and join the points so found. Then the perpendicular from t_1 upon this joining line will be the required tangent.

6. If we take four points on the curve so as to form a parallelogram, we can construct four inscribed triangles from these four points. The four orthocentres of these triangles will again form a parallelogram.

If we take four points on the hyperbola so as to form an isosceles trapezoid, the orthocentres formed from the four inscribed triangles will again form an isosceles trapezoid. Taking these last four points and repeating the process, we get the four original points; that is, one may say that an isosceles trapezoid inscribed in an equilateral hyperbola has a periodicity two.

YALE UNIVERSITY, JANUARY, 1903.

*Cf. problem 501 in C. Taylor's *Geometry of Conics* (1881), p. 184: "In any right angled triangle inscribed in an equilateral hyperbola the perpendicular upon the hypothenuse is the tangent at the right angle."

A NEW PROOF OF THE GENERALIZED WILSON'S THEOREM

By G. A. Miller

By employing several elementary theorems of substitution groups it is possible to give a very simple proof of Wilson's theorem in the theory of numbers. This proof makes no use of the theory in regard to the number of roots of a congruence. In fact, the idea of congruence is not employed at all except in so far as this concept is involved in that of periodicity. We proceed to develop the theorems of substitution groups which will be employed.

1. Statement of the Theorem. Product of All the Operators in an Abelian Group. Wilson's theorem relates to the $\phi(g)$ numbers prime to an integer g and less than g, and asserts that their product is congruent, *modulo* g, to -1 in three cases, viz. when $g = p^a$, $2p^a$, or 4, and in all other cases the product is congruent to $+1$. (Here p denotes an odd prime, and a any positive integer). Now these $\phi(g)$ numbers combined by multiplication and the products taken *modulo* g form a group of a certain special kind, the group of isomorphisms of a cyclic group with itself. The order of the cyclic group is g, and the group of isomorphisms is necessarily abelian. Further details have been explained in a recent number of the Annals (Second Series, Vol. 2, p. 77). We shall expect, therefore, to come upon the equivalent in Group theory of Wilson's theorem in numbers if we examine abelian groups, particularly such as can be groups of isomorphisms of cyclic groups.

All the operators whose orders exceed two in any group may be so arranged that their continued product is the identity. This may be done, for instance, by associating each operator with its inverse. As in an abelian group (A) any product is independent of the order of the factors, it follows that *the continued product of all the operators whose orders exceed two in A is the identity.*

If the group A contains no operator of order two, then the product of all its operators is the identity. It remains to examine separately the operators of order two when any such are contained in the group.

All the operators of order two in A generate a group (A_1) of order 2^a, which has a independent generators of order 2. If s_1 denotes any one of these, then A_1 is the direct product of the group $(1, s_1)$ and the subgroup of order

$2^{\alpha-1}$ generated by the remaining independent generators. It follows that s_1 is a factor of one-half of the operators of A_1 and hence does not appear in the continued product of all the operators of A_1 whenever $\alpha > 1$. Since any operator of order 2 could have been taken for s_1, we have the theorem: *If an abelian group contains more than one operator of order two, the continued product of all its operators is the identity; if it contains only one such operator, this operator is the continued product of all its operators.*

2. Order of a Cyclic Group whose Group of Isomorphisms Contains only One Operator of Order Two. We proceed to prove that the order of such a cyclic group is of one of the three forms, p^α, $2p^\alpha$, or 4. Let M represent any metacyclic substitution group of order $p(p-1)$ and of degree p. The subgroup (M_1) which is composed of all the substitutions of M which omit a given letter is regular and of degree $p-1$, since each one of its substitutions transforms each substitution of order p in M into a power of itself. As M_1 is a group of isomorphisms of a cyclic group, it is abelian.* If it contained more than one substitution of order 2, M would contain more than p such substitutions, since no two conjugates under M can occur in M_1 and each substitution of order 2 in M has p conjugates under M.

Suppose, then, that M could contain more than p substitutions of order 2. Each of these would involve $(p-1)/2$ transpositions, and, as there are only $p(p-1)/2$ distinct transpositions of p letters, two substitutions of order two would have a common transposition. Their product would, therefore, be of a lower degree than $p-1$. As this is contrary to the fact that M_1 is regular and of degree $p-1$, it follows that *the group of isomorphisms of a cyclic group of order p contains only one operator of order* 2.

The group of isomorphisms of the cyclic group of order p^α, $\alpha > 1$, is the direct product of the cyclic group of order $p^{\alpha-1}$ and of the group of isomorphisms of the group of order p.† From this it follows that the group of isomorphisms of the cyclic group of order p^α (and hence also of the cyclic group of order $2p^\alpha$) contains only one operator of order 2. It has been proved without the use of the theory of congruences† that the group of isomorphisms of the cyclic group of order 2^α contains just three operators of order 2 whenever $\alpha > 2$; when $\alpha = 2$ this group of isomorphisms is evidently of order 2. That is,

* *Transactions of the American Mathematical Society*, vol. 1 (1900), p. 397. From the theory of primitive roots it follows directly that M_1 is the cyclic group of order $p-1$, but the present proof is independent of this theory.

†*Bulletin of the American Mathematical Society*, vol. 7, 1901, p. 350.

if the group of isomorphisms of a cyclic group of order 2^a contains only one operator of order 2, then a must be 2.

Since the group of isomorphisms (G_1) of the cyclic group (G) of order $g = 2^{a_0} p_1^{a_1} p_2^{a_2} \cdots p_m^{a_m}$ is the direct product* of the groups of isomorphisms of the cyclic groups of orders $2^{a_0}, p_1^{a_1}, p_2^{a_2}, \cdots p_m^{a_m}$, and since each of these groups of isomorphisms is of even order (except that of order 2^{a_0} when $a_0 = 1$) it follows that G_1 must involve more than one operator of order 2 with the exception of the three cases when its order is p^a, $2p^a$, or 4 where p is any odd prime. That is, *if the group of isomorphisms of a cyclic group contains only one operator of order 2, the order of the cyclic group must be one of the three numbers 4, p^a, $2p^a$, where p is any odd prime.*

3. Conclusion. As stated in the introduction, the group G_1 of the preceding paragraphs is representable by the $\phi(g)$ numbers less than g and prime to g, when they are combined by multiplication, the products being replaced by their least residues *modulo* g. To complete now the proof of the generalized Wilson's theorem it is only necessary to notice that in the three exceptional cases specified in §1 the one operator of order two corresponds to the numbers $p^a - 1$, $2p^a - 1$, and 3 respectively,—numbers congruent in their respective systems to -1, while the identity corresponds to $+1$. With these facts in mind it is evident that the theorems demonstrated in §§1, 2 constitute precisely a statement, in the language of group theory, of Wilson's theorem.

STANFORD UNIVERSITY, DECEMBER, 1902.

* *Transactions of the American Mathematical Society*, vol. 1, 1900, p. 396, Theorem II.

A SUFFICIENT CONDITION FOR THE MAXIMUM NUMBER OF IMAGINARY ROOTS OF AN EQUATION OF THE N-TH DEGREE

By E. B. Van Vleck

So far as I have been able to ascertain, the following theorem is new:

If c_i is real and the terms of the sequence

$$c_0, \begin{vmatrix} c_0 & c_1 \\ c_1 & c_2 \end{vmatrix}, \begin{vmatrix} c_0 & c_1 & c_2 \\ c_1 & c_2 & c_3 \\ c_2 & c_3 & c_4 \end{vmatrix}, \cdots\cdots, \Delta_n = \begin{vmatrix} c_0 & c_1 & c_2 & \cdots & c_n \\ c_1 & c_2 & c_3 & \cdots & c_{n+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ c_n & c_{n+1} & c_{n+2} & \cdots & c_{2n+2} \end{vmatrix}, \tag{1}$$

are positive, all of the roots of the equation

$$c_0 + c_1 x + c_2 x^2 + \cdots\cdots + c_{2n} x^{2n} = 0 \tag{2}$$

are imaginary, and all but one of the roots of the equation

$$c_0 + c_1 x + c_2 x^2 + \cdots\cdots + c_{2n+1} x^{2n+1} = 0.* \tag{3}$$

An extremely simple proof of this result can be obtained with the aid of a theorem of Sylvester† concerning the minors of Δ_n which I will state in the following restricted form: when Δ_n and the minors appearing in (1) are all positive, any minor formed symmetrically from Δ_n by erasing corresponding rows and columns will also be positive. In particular,

$$c_{2i} > 0, \qquad \begin{vmatrix} c_{2i} & c_{2i+1} \\ c_{2i+1} & c_{2i+2} \end{vmatrix} > 0. \tag{4}$$

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* This is somewhat like the well known theorem given by Borchardt in *Liouville's Journal*, vol. 12 (1846), p. 58. The latter uses a similar sequence of determinants constructed with s_i (the sum of the i-th powers of the roots) in place of c_i and continuing for nearly twice as many terms until the discriminant is reached. The number of pairs of imaginary roots is equal to the number of changes of sign in the sequence. Thus $2n$ roots will be imaginary if, and only if, there are n changes of sign. This test for the maximum number of imaginary roots has the advantage of being a necessary one, but, on the other hand, it is much less convenient in its application since it not only involves a greater number of terms but also necessitates the preliminary computation of the s_i.

†*Philosophical Transactions*, vol. 143 (1853), p. 482, foot-note. See also *Philosophical Magazine*, Ser. 4, vol. 4 (1852), p. 140-141.

Now we have identically

$$c_0 + c_1x + c_2x^2 + \cdots + c_{2n}x^{2n} = c_0\left(1 + \frac{c_1}{2c_0}x\right)^2 + \frac{c_0c_2 - c_1^2}{4c_0}x^2$$
$$+ \frac{3c_2x^2}{4}\left(1 + \frac{2}{3}\frac{c_3}{c_2}x\right)^2 + \frac{2}{6}\frac{c_2c_4 - c_3^2}{c_2}x^4$$
$$+ \frac{4c_4x^4}{6}\left(1 + \frac{3}{4}\frac{c_5}{c_4}x\right)^2 + \frac{3}{8}\frac{c_4c_6 - c_5^2}{c_4}x^6$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$+ \frac{(n+1)c_{2n-2}x^{2n-2}}{2n}\left(1 + \frac{n}{n+1}\frac{c_{2n-1}}{c_{2n-2}}x\right)^2 + \frac{n}{2(n+1)}\frac{c_{2n-2}c_{2n} - c_{2n-1}^2}{c_{2n-2}}x^{2n}$$
$$+ \frac{n+2}{2(n+1)}c_{2n}x^{2n}.$$

Since the right hand member consists of the sums of squares multiplied into positive coefficients, the roots of (2) must be imaginary.

To prove the second part of the theorem, put $x = 1/z$ and write equation (3) in the form

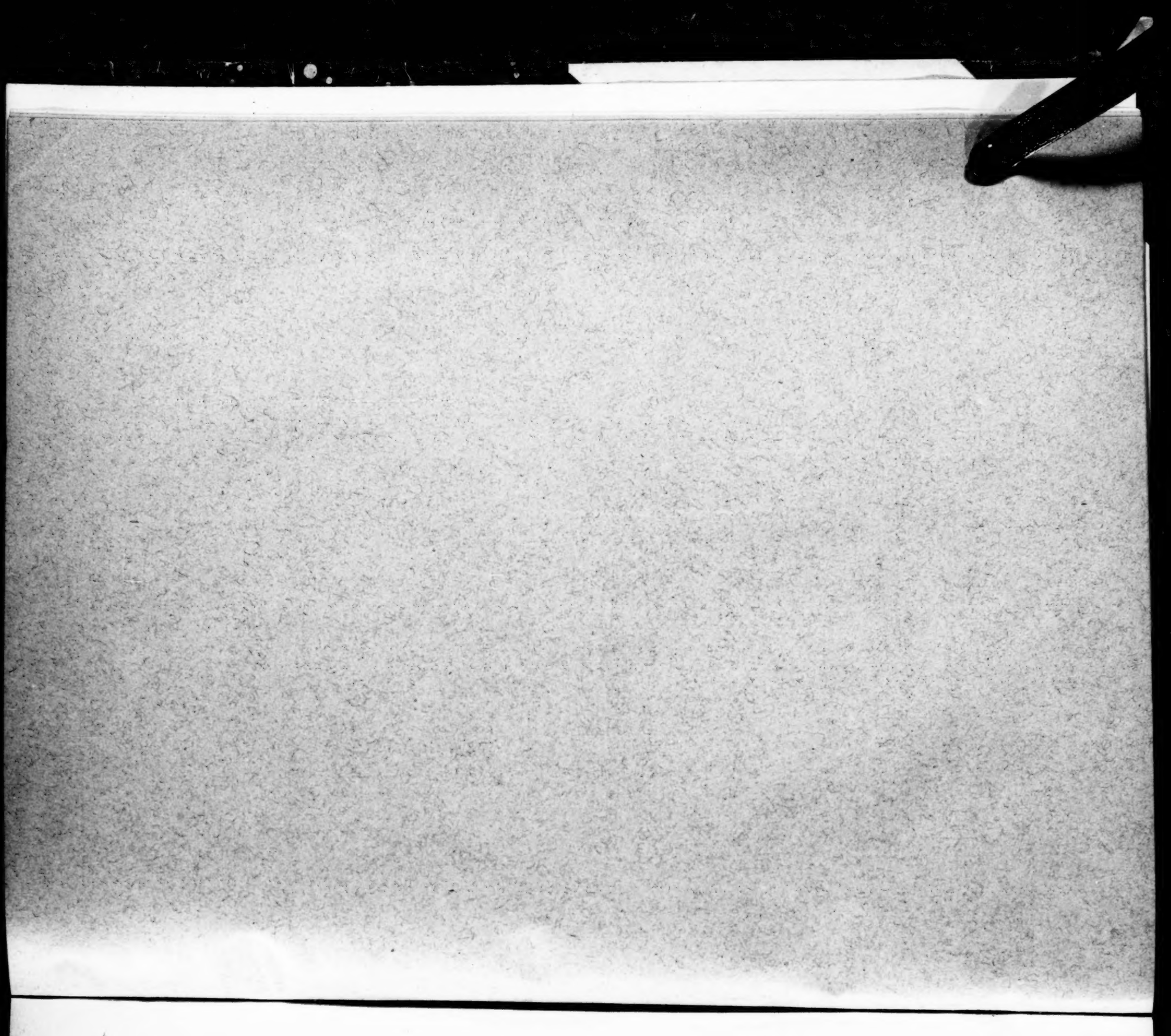
$$c_0z^{2n+1} + c_1z^{2n} + \cdots + c_{2n}z + c_{2n+1} = 0. \qquad (5)$$

The derivative of the left hand member can be expressed as

$$c_{2n}\left(1 + \frac{c_{2n-1}}{c_{2n}}z\right)^2 + \frac{c_{2n-2}c_{2n} - c_{2n-1}^2}{c_{2n}}z^2$$
$$+ 2c_{2n-2}z^2\left(1 + \frac{c_{2n-3}}{c_{2n-2}}z\right)^2 + 2\frac{c_{2n-4}c_{2n-2} - c_{2n-3}^2}{c_{2n-2}}z^4$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$+ nc_2z^{2n-2}\left(1 + \frac{c_1}{c_2}z\right)^2 + n\frac{c_2c_0 - c_1^2}{c_2}z^{2n} + (n+1)c_0z^{2n}.$$

But again, by (4), this is the sum of positive terms. Since the derivative of the left hand member of (5) is positive, it must be an increasing function of z and so vanishes but once when z traverses the real axis.

WESLEYAN UNIVERSITY, MAY, 1903.



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